Maximum Efficiency of Direct-Radiator Loudspeakers 3193 (G-3)

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The Thiele/Small method of low-frequency direct-radiator loudspeaker system analysis neglects the radiation impedance components in the equivalent electric network model. When these components are included some surprising results are evident. Due to the definition of efficiency widely used in direct-radiator loudspeaker analysis, the absolute maximum efficiency is limited to 25%. With voice-coil inductance neglected, inclusion of the radiation impedance components transform all responses from high-pass into band-pass functions. For low frequencies, the maximum achievable nominal power transfer efficiency is found to be proportional to cone diameter. For a specific diameter, the maximum efficiency depends only on the moving mass to air-load mass ratio. Relationships and graphs are presented which relate the true nominal power-transfer efficiency to the Thiele/Small derived efficiency.

0. GLOSSARY OF SYMBOLS

a — effective radius of driver diaphragm

B — magnetic flux density in driver air gap

β — ratio of of driver voice coil dc resistance to electrical resistance representing the real part of the acoustic radiation load ($= R_E / R_{EAR} = R_1 / R_2$ $= 128\rho_0 c/9\pi^2 \cdot S_D R_E/B^2l^2 \approx 1879\, a^2R_E/B^2l^2$)

c — velocity of sound in air (= 343 m/s at 20° C)

C_{AT} — total acoustic compliance of driver and enclosure

C_{EAR} — electrical capacitance representing air-load mass on driver diaphragm ($= M_{MA} / B^2l^2$) (for $ka < 0.3$, $\approx (8\rho_0 / 3) \cdot a^3 / B^2l^2 \approx 3.23\, a^3 / B^2l^2$)

C_{MES} — electrical capacitance representing driver moving mass ($= M_{MD} / B^2l^2$, not including air-load mass)

C_1 — pseudonym for C_{MES}

C_2 — pseudonym for C_{EAR}

d — effective diameter of driver diaphragm ($= 2a$)

e_{in} — input voltage applied to terminals of loudspeaker, V rms

e_{out} — output voltage appearing across real part of radiation load in circuit model (R_{EAR})

E — error value used in calculation of Thiele-Small Efficiency vs true efficiency

f — natural frequency variable in Hz

f_0 — resonance frequency of second-order band-pass system (frequency at peak of response)

$G(s)$ response function ($= e_{out}(s)/e_{in}(s)$)

k wave number ($= \omega / c = 2\pi / \lambda$)

l length of voice-coil conductor in magnetic gap

L_{CET} electrical inductance representing total system compliance ($= C_{AT}B^2l^2 / S_D^2$)

L_E electrical inductance of driver voice coil

M_{MA} mechanical air-load mass on driver diaphragm ($= 1 / \omega Y_I$), note that this value varies with frequency and can be approximated at low frequencies by $8\rho_0 a^3 / 3 \approx 3.23\ a^3$ (for $ka < 0.3$)

M_{MD} mechanical moving mass of driver diaphragm assembly <u>not</u> including air load

M_{MS} mechanical moving mass of driver diaphragm assembly including air load ($= M_{MD} + M_{MA}$)

P_A acoustic output power ($= e_{out}^2 / R_{EAR}$ in circuit model)

P_E nominal electrical input power ($= e_{in}^2 / R_E$)

Q ratio of reactance to resistance (series circuit) or resistance to reactance (parallel circuit)

Q_0 Q of second-order band-pass system at peak of response (f_0)

R_{AS} acoustic resistance of driver suspension losses

R_E dc resistance of driver voice coil

R_{EAR} electrical resistance representing the real part of the acoustic radiation load ($= B^2l^2 Y_R$), note that this value varies with frequency and can be approximated at low frequencies by $(9\pi^2 / 128) \bullet B^2l^2 / \rho_0 c S_D = B^2l^2 / 1879a^2$ (for $ka < 0.3$)

R_{ES} electrical resistance representing driver mechanical losses ($= R_{AS}B^2l^2 / S_D^2$)

R_1 pseudonym for R_E

R_2 pseudonym for R_{EAR}

s complex frequency variable ($= \sigma + j\omega$)

S_D effective surface area of driver diaphragm ($= \pi a^2$)

η efficiency, (= power out / power in)

η_0 asymptotic reference efficiency of driver defined as per R. E. Small (known as the Thiele/Small reference efficiency) ($= (\rho_0 / 2\pi c) \bullet B^2l^2 / R_E \bullet S_D^2 / M_{MS}$)

η_0' same as η_0, except that efficiency is computed using the moving mass only (M_{MD}), which does not include the air-load mass

η_{max} maximum efficiency

η_{peak} efficiency of second-order band-pass system at peak of response (f_0)

ρ_0 density of air ($= 1.21\ kg/m^3$)

ω radian frequency variable ($= 2\pi f$)

ω_0 radian center frequency of second-order bandpass filter

ω_1	radian corner frequency corresponding to driver moving mass -- driver voice-coil resistance ($= 1 / R_1C_1 = B^2l^2/(R_EM_{MD})$
ω_2	radian corner frequency corresponding to the frequency where the radiation reactance and resistance are equal at about $ka = 1$ ($= 1 / R_2C_2 = 1 / R_{EAR}C_{EAR} = (16/3\pi)\ c/a \approx 1.7\ c/a$))
ω_N	normalized frequency ($= \omega / \omega_2 = \omega R_1C_1$)
Y_M	complex mechanical radiation mobility (reciprocal of mechanical radiation impedance) of the air load upon one side of a plane piston of radius a mounted in an infinite flat baffle ($= Y_R + \mathrm{j}\ Y_I$)
Y_R	real part of the mechanical radiation mobility of the air load upon one side of a plane piston of radius a mounted in an infinite flat baffle, note that this value varies with frequency and can be approximated at low frequencies ($ka < 0.3$) by $(9\pi^2 / 128) / \rho_0 c S_D$
Y_I	imaginary part of the mechanical radiation mobility of the air load upon one side of a plane piston of radius a mounted in an infinite flat baffle, note that this value varies with frequency and can be approximated at low frequencies ($ka < 0.3$) by $(3\pi / 8) / \rho_0 c S_D ka$
ψ	ratio of driver moving mass to air-load mass ($= M_{MD} / M_{MA} = C_{MES} / C_{EAR} = C_1 / C_2 = M_{MD} / (8\rho_0 a^3/3) = 3/(8\rho_0) \cdot M_{MD} / a^3 \approx 0.31\ M_{MD} / a^3$)

1. INTRODUCTION

The widely used methods of direct-radiator loudspeaker system analysis, based on the pioneering work of Thiele and Small, neglect the radiation impedance components in deriving the system response functions [1, 2, 3, 4]. Neglecting these components provides a very powerful simplification of the equivalent circuit that helps the designer in deriving the appropriate response functions and system relationships. One of the main effects of not including the radiation terms in the analysis, is that all derived responses are high-pass functions. Another effect is the illusory masking of the efficiency relationships in concealing how the functions behave at high efficiencies (assumed here to be efficiencies greater than about 5 to 10%). Questions such as: will the efficiency continue to double (+3 dB) each time I raise the *Bl* product by the square root of two?, will the efficiency continue to double each time I double the number of units in an array?, or how high can I really get the system efficiency by combining multiple units in arrays?, are not answered.

Beranek [5, p. 183], although not neglecting the radiation components in his analysis, does not provide any systematic evaluation of the resultant model except for isolated examples. Locanthi [6] in an excellent early work, which applies electric circuit analogs to loudspeaker design, calculates a transmission coefficient and a maximum theoretical efficiency for the specific situation of a driver having a very high *Bl* product, but does not extend the analysis to the general case where an arbitrary *Bl* product sets the mid-band efficiency.

In this study, I include the radiation impedance components in the model and derive relationships that yield a number of important relationships. These include: the absolute maximum efficiency, the maximum efficiency as a function of radiator size and frequency, the maximum efficiency as a function of moving mass to air-load mass ratio, the true nominal power transfer efficiency as a function of frequency with comparison to the Thiele/Small efficiency, and derive the value of the *Bl* product that maximizes efficiency for a particular set of driver parameters.

One important observation, of this study, is the limitation of the efficiency to an absolute maximum of 25%. This is due directly to the specific definition of efficiency that is widely used in direct-radiator loudspeaker system analysis. In this definition, the input power, rather than being the true input power, which varies with frequency depending on the impedance of the speaker, is a fictitious power called the nominal electrical input power, which does not change with frequency. In this definition of efficiency, called the nominal power transfer ratio, the efficiency is the ratio of the actual system acoustic output power to the fictitious electrical input power delivered into a fixed resistance by the same source. This fixed resistance is assumed to be the dc resistance of the driver voice coil [2]. If the fixed resistance were equal to twice the dc resistance of the driver voice coil, as is done in horn analysis, the absolute maximum efficiency would be 50%, rather than 25%. In practical measurement situations, the fixed resistance is often assumed to be equal to the minimum impedance within the defined operating bandwidth. This minimum impedance is sometimes 25% greater than the dc resistance, and as a consequence, the efficiency might be 1 dB greater than the value measured using the dc resistance. Furthermore, if the bandwidth of operation is restricted to a narrow range around the system resonance, the impedance in this range may be even higher, thus raising effective efficiency even further. This paper, however, will define the fixed resistance as being equal to the dc resistance, in keeping with the precedent of past theoretical analysis.

The assumption that the input power is equal to the nominal input power greatly simplifies the speaker system analysis and makes the efficiency frequency response equal to the frequency response that you measure under actual use situations, ie driving the system with a constant voltage source. If the true input power were used instead, the efficiency frequency response would differ greatly from the in-use frequency response, for the majority of systems.

Mutual acoustic radiation impedance and coupling of an array of similar radiators is not an issue in this analysis [7]. What is being analyzed, in every case, is a single circular radiator of arbitrary dimensions. If one prefers to think of the single radiator as being composed of an array of smaller elements, each sub-element radiator must be shaped in such a way as to achieve a 100% packing density, (perhaps a square sub-element radiator). This situation corresponds to 100% mutual coupling and makes the array of elements completely equivalent to the larger radiator, of the same outside dimensions, that it replaces.

2. REVIEW OF GENERALIZED DIRECT-RADIATOR LOUDSPEAKER MODEL ANALOGOUS CIRCUIT

In this paper, all analogous circuits are of the mobility type with all electrical quantities referred to the electrical side [5]. In the mobility-type analogous circuit (dual of the impedance-type analogous circuit), voltage is the analog of velocity (or volume velocity) and current is analogous to force (or pressure). In this analog, an acoustic or mechanical mass is represented by a capacitor, and an acoustic or mechanical compliance is represented by an inductor.

2.1 Complete Model

Fig. 1 shows a relatively complete analogous circuit model of a direct-radiator loudspeaker mounted in a closed-box enclosure. Components represented include: dc resistance of voice coil (R_E), inductance of voice coil (L_E), total system compliance (including driver suspension compliance and possible rear air cavity compliance) (L_{CET}), moving mass (mass of driver diaphragm assembly including voice coil) (C_{MES}), mechanical losses (R_{ES}), and acoustic radiation components (R_{EAR} and C_{EAR}). The acoustic radiation components are represented here by a capacitor and resistor whose values vary with frequency. It will be shown later that these components can be assigned constant non-frequency-varying values without effecting the operation of the circuit to any great extent..

2.2 Simplified Model

For analysis purposes in this paper, a simplified version of the circuit model of Fig. 1 was generated by removing or short-circuiting appropriate elements, the radiation components are of course left in. This simplified circuit model is shown in Fig. 2. Two elements of the model were neglected: voice inductance (L_E) by short-circuiting, and driver mechanical losses (R_{ES}) by removal (only driver electro-mechanical damping and damping due to sound radiation is included). For the analysis in this paper, various configurations of the simplified circuit were evaluated to derive various maximum efficiency relationships.

3. EFFICIENCY

The efficiency relationships used here are related directly to component values in the analogous electrical circuit model, rather than starting from the original acoustical definitions (the relationships are exactly equivalent however). This means that the acoustic output power is that power developed in the circuits electrical resistance that represents the real part of the acoustic radiation load.

3.1 Definition

The definition of efficiency used here is essentially the same as Small's [2], except that the output resistance of the source is neglected ($R_g = 0$, in Small's model). The efficiency or nominal power transfer ratio is defined as being the ratio between the radiated acoustic power (P_A) and the nominal electrical input power (P_E):

$$\eta = \frac{P_A}{P_E} . \tag{1}$$

3.2 Nominal Electrical Input Power

The nominal electrical input power is defined here as the power delivered by the source, with zero source resistance, into a resistor having the same value as the driver voice coil resistance (R_E):

$$P_E = \frac{e_{in}^2}{R_E} . \tag{2}$$

As will be shown later, it is this widely used definition of input power that constrains the absolute maximum efficiency of direct-radiator loudspeakers to 25%.

3.3 Acoustic Output Power

The radiated acoustic output power is defined here as the power delivered by the circuit model into the resistance that represents the real part of the acoustic radiation load:

$$P_A = \frac{e_{out}^2}{R_{EAR}} . \tag{3}$$

3.4 Acoustic Radiation Load

The acoustic load for this study is assumed to be that of the air load on one side of a plane circular piston mounted in an infinite flat baffle radiating sinusoidally. [8, p. 179], [9, p. 326]. The load can be looked at in two different ways: as a mechanical (or acoustic) impedance where voltage is the analog of force (or pressure) and current is analogous to velocity (or volume velocity), or as a mechanical (or acoustic) mobility, which is the reciprocal of impedance (equivalent to admittance in electrical circuits), where voltage is the analog of velocity (or volume velocity) and current is analogous to force (or pressure). In the electrical analogous circuit, the mobility load is the proper one to use.

3.4.1. Acoustic Impedance

Without going into any detail about the functions which define the acoustic load on one side of a plane circular piston mounted in an infinite flat baffle [5],[8], and [9], Fig. 3 shows the familiar normalized real and imaginary parts of the radiation impedance functions for this situation. The analogous circuit, that corresponds to this radiation impedance, is a parallel RL circuit whose component values are not constant and vary with frequency.

3.4.2. Acoustic Mobility (Admittance)

Fig. 4 shows the corresponding real and imaginary parts of the normalized radiation mobility functions for the same situation. The analogous circuit that corresponds to the radiation mobility is a series RC circuit whose component values are not constant and are functions of frequency. As can be seen from the graph, for frequencies below the frequency where $ka = 1$ ($f = c\,/\,2\pi a$) these functions approach straight lines. The correspondence of these functions to the reactance and resistance of a series RC circuit is quite clear.

3.4.3. Approximate Analogous Circuit

Below $ka = 0.3$, the normalized radiation mobility functions can be approximated by the following asymptotic functions [5]:

$$\text{real part} = Y_R \bullet \rho_0 c S_D = \frac{9\pi^2}{128} \approx 0.694 \text{, and} \tag{4}$$

$$\text{imaginary part} = Y_I \bullet \rho_0 c S_D = \frac{3\pi}{8ka} \approx \frac{1.18}{ka}\,. \tag{5}$$

When these values are transferred over to the electrical side of the model, this load corresponds to a series RC high-pass circuit whose component values are:

$$R_2 = R_{EAR} = B^2 l^2\, Y_R = \frac{9\pi^2}{128} \bullet \frac{B^2 l^2}{\rho_0 c S_D} \approx \frac{B^2 l^2}{1879 a^2} \text{, and} \tag{6}$$

$$C_2 = C_{EAR} = \frac{1}{B^2 l^2 \omega Y_I} = \frac{ka\rho_0 c S_D}{B^2 l^2 \frac{3\pi}{8}} = \frac{8\rho_0}{3} \bullet \frac{a^3}{B^2 l^2} \approx \frac{3.23 a^3}{B^2 l^2}\,. \tag{7}$$

Equating Eqs. (4) and (5) and solving for *ka* yields:

$$ka = \frac{16}{3\pi} \approx 1.7 \tag{8}$$

This value is the corner frequency at which the R and C values of Eqs. (6) and (7) correspond:

$$\omega_2 = \frac{1}{R_{EAR}C_{EAR}} = \frac{1}{R_2 C_2} = \frac{16}{3\pi} \bullet \frac{c}{a} \approx 1.7\frac{c}{a} \tag{9}$$

Note that this corner frequency is not at the frequency where $ka = 1$, but is roughly 70% higher. Further on in this study, the normalized value of omega $\omega_N = \omega / \omega_2$ will be used. The dimensionless value ω_N can be giving in terms of the dimensionless value ka, as follows:

$$\omega_N = \frac{3\pi}{16} ka \approx 0.589 ka \approx ka / 1.7 \tag{10}$$

3.5 Efficiency vs Frequency

The previous values for nominal electrical input power (P_E), Eq. (2) and acoustic output (P_A), Eq. (3), may be used in Eq. (1) to yield the general equation for efficiency vs frequency in terms of the response function of the network, *G(jω)*:

$$\eta(\omega) = \frac{P_A}{P_E} = \frac{\dfrac{e_{out}^2}{R_{EAR}}}{\dfrac{e_{in}^2}{R_E}} = \left|\frac{e_{out}}{e_{in}}(j\omega)\right|^2 \frac{R_E}{R_{EAR}} = |G(j\omega)|^2 \frac{R_E}{R_{EAR}} = |G(j\omega)|^2 \frac{R_1}{R_2} \tag{11}$$

4. MAXIMUM EFFICIENCY

The maximum efficiency for several different conditions of the simplified circuit model of Fig. 2 were analyzed. These conditions included: 1. the absolute maximum efficiency when only the driver dc voice coil resistance and the real part of the radiation load are considered (infinite system compliance, zero moving mass and infinite air load mass), 2. the maximum efficiency with infinite system compliance and zero moving mass, and 3. the maximum efficiency with infinite system compliance.

4.1 Absolute Maximum Efficiency With Infinite Suspension Compliance, Zero Moving Mass, and Infinite Air Load Mass

This situation corresponds to the condition where all the components that might impede the flow of power from source to load, have been removed. Only the resistance of the voice coil and the real part of the radiation load remain, forming a voltage divider.

4.1.1. Model

The model for the situation of infinite suspension compliance, zero moving mass, and infinite air-load mass is shown in Fig. 5. This model is the simplest possible case and just includes the driver dc voice coil resistance ($R_E = R_1$) connected directly to the radiation load ($R_{EAR} = R_2$). The driver diaphragm area and Bl product are the only parameters that can be adjusted to set the relative size of the reflected radiation resistance as compared to R_E. Adjusting these parameters raises and lowers the efficiency, but as will be shown, the efficiency can only be raised so far.

4.1.2. Response Function

The response function for this situation is independent of frequency and is just equal to the voltage divider action of the two resistors:

$$G(j\omega) = \frac{R_2}{R_1 + R_2} = \frac{R_{EAR}}{R_{EAR} + R_E} \qquad (12)$$

4.1.3. Efficiency Function

Combining Eqs. (11) and (12) yields:

$$\eta(\omega) = \left(\frac{R_2}{R_1 + R_2}\right)^2 \left(\frac{R_1}{R_2}\right) = \frac{R_1 R_2}{(R_1 + R_2)^2} = \frac{\dfrac{R_1}{R_2}}{\left(\dfrac{R_1}{R_2} + 1\right)^2} = \frac{\beta}{(\beta + 1)^2}, \qquad (13)$$

where

$\beta = R_1 / R_2.$

Fig. 6 shows a plot of this relationship as a function of β.

4.1.4. Absolute Maximum Efficiency

The graph of Fig. 6 shows a maximum of 0.25 (25%) at $\beta = 1$, which is equivalent to $R_1 = R_2$ or $R_E = R_{EAR}$. This result is verified by finding the maximum of Eq. (12) with respect to β, by differentiating Eq. (12) with respect to β, equating to zero, and solving for β and then substituting back in Eq. (12). When this is done, the maximum occurs at $\beta = 1$, where the efficiency is $1/(1+1)^2 = 1/4 = 0.25 = 25\%$.

This relatively low value of absolute maximum efficiency (rather than 50% which would be expected for a load connected to a generator with the load resistance equal to the generator's source resistance) is a direct result of the definition of nominal electrical input power used in the definition of efficiency. If the input resistance of the driver were defined as being $2R_E$ rather than R_E, as it is in the definition of efficiency for horn loaded drivers [10], the maximum efficiency would be 50% rather than 25%, as it is here. Alternately, efficiency could be defined as relating true output power versus true input power (rather than nominal input power), which would remove this limitation and would allow efficiency to rise to 100%. Unfortunately, this would mean that the efficiency frequency response function would differ from the response function measured with constant input voltage, and thus the efficiency frequency response would differ greatly from the response under normal operating conditions. As an example, the true efficiency of a closed box system at box resonance can approach 100%, because the input impedance could be very high at resonance, thus making the input power quite low. This would reflect as a peak in the efficiency frequency response that would not be representative of the frequency response as the system is normally used, with a constant voltage drive source.

This power transfer condition is exactly the same as a generator having a source resistance of R_1 driving a load of resistance R_2. Maximum power is transferred when the load resistance is equal to the source resistance. In this case, however, the efficiency is reduced by a factor of 2, because the input power is twice as high due to the definition of nominal electrical input power.

4.2 Efficiency With Infinite Suspension Compliance And Zero Moving Mass

This situation differs from the previous situation in that a finite radiation air-mass load is included in the model. The power, in effect, has to flow through the radiation air mass (capacitor) to get to the load. In the mobility model, an infinite air mass (as in the last situation analyzed) provides no impediment to power transmission. A finite air mass, however provides more impediment to power flow the lower the frequency. The frequency at which the power starts to become limited is approximately the frequency where $ka \approx 1$ or $\omega_2 \approx 1.7\ c/a$. At all lower frequencies, the impediment increases in direct proportion to the decrease in frequency. Thus, the main effect of finite radiation air mass, is a rolloff of efficiency at the rate of 6 dB per octave below the frequency where $ka \approx 1$.

4.2.1. Model

The model for the situation is the same as the last except for the addition of a capacitor C_2, in series with the load, corresponding to the air mass component of the radiation load. The circuit model corresponding to this situation is shown in Fig. 7. Be aware that the values of both the resistor (R_2) and capacitor (C_2), that model the complex radiation load, are non-linear and vary as a function of frequency. As was shown before, however, approximate constant values may be chosen that mimic the behavior of the actual load throughout the low-frequency range ($ka < 1$). Note that corner frequency ω_1, where the reactance of C_2 equals R_2, is fixed once the the frequency that corresponds to $ka = 1$ has been chosen. The Bl product and cone area parameters can be changed to scale the relative impedance levels of the radiation load as compared to the resistance of the voice coil (R_E).

4.2.2. Response Function

If the approximate analogous circuit is used for the radiation load, the response function for this situation is a first-order high-pass filter. The level at high-frequencies is set by the voltage divider ratio between R_1 and R_2, with a corner frequency of $\omega = 1 / ((R_1 + R_2)C_2)$ where the response is 3 dB down from the plateau level of $R_2/(R_1 + R_2)$. The response function for this situation appears as:

$$G(s) = \frac{\left(\dfrac{R_2}{R_1 + R_2}\right)s}{s + \dfrac{1}{(R_1 + R_2)C_2}} \tag{14}$$

4.2.3. Efficiency Function

Eq. (14) may be converted to magnitude-squared form and appears as follows:

$$|G(j\omega)|^2 = \frac{\left(\dfrac{R_2}{R_1 + R_2}\right)^2 \omega^2}{\omega^2 + \left(\dfrac{1}{(R_1 + R_2)C_2}\right)^2} \tag{15}$$

Combining Eqs. (11) and (15) yields:

$$\eta(\omega) = |G(j\omega)|^2 \frac{R_1}{R_2} = \frac{\dfrac{R_1 R_2}{(R_1 + R_2)^2}\omega^2}{\omega^2 + \dfrac{1}{(R_1 + R_2)^2 C_2{}^2}} \tag{16}$$

Eq. (16) may be rewritten in normalized form by choosing $\beta = R_1 / R_2$, $\omega_2 = 1 / (R_2C_2)$, and $\omega_N = \omega / \omega_2$:

$$\eta(\omega) = \frac{\dfrac{\beta}{(\beta+1)^2}\omega_N{}^2}{\omega_N{}^2 + \dfrac{1}{(\beta+1)^2}} \tag{17}$$

In various frequency regions the efficiency response takes on the following values:

for $\omega_N >> \dfrac{1}{\beta+1}$

(high frequencies)

$$\eta(\omega_N) = \frac{\beta}{(\beta+1)^2}\text{, the highest plateau level,} \tag{18}$$

for $\omega_N = 1$

$$\eta(1) = \frac{\beta}{\beta^2 + 2\beta + 2}, \tag{19}$$

for $\omega_N = \dfrac{1}{\beta+1}$

(the corner frequency)

$$\eta\left(\frac{1}{\beta+1}\right) = \frac{1}{2} \bullet \frac{\beta}{(\beta+1)^2}\text{, one-half the HF plateau level,} \tag{20}$$

for $\omega_N << \dfrac{1}{\beta+1}$

(low frequencies)

$$\eta(\omega_N) = \beta\,\omega_N{}^2\text{, rolls off at 6-dB per octave.} \tag{21}$$

4.2.4. Families of Curves

Eq. (17) was used to plot several efficiency response curves for various values of β ($\beta >=$ 1) that yielded specific plateau efficiency values. This family of curves is shown in Fig. 8. Observe that as the plateau efficiency level decreases, each curve goes down lower in frequency at the half-power points (Eq. 20). The maximum efficiency versus *ka*, computed in the next section, effectively is a locus of the half-power response points of the many individual curves. Note that each individual curve rolls off at 6-dB per octave below its half-power point but that the locus of the half-power points rolls off at only 3-dB per octave.

It must be realized that these curves were plotted from a function that used the RC approximate values for the radiation load (Eqs. (6) and (7)). Fig. 9 shows a repeat of Fig. 8 except that the actual radiation-load mobility values of Fig. 4, were used. In this case, β was assigned values that made the efficiency plateau values, for the $ka < 1$ frequency range, equal to the chosen efficiencies in Fig. 8. The main effect of using the actual load values is an increase of efficiency by about 20% above $ka = 2$, and a slight ripple on the response curves.

4.2.5. Maximum Efficiency vs *ka* and wavelength

In this section I calculate the upper efficiency limit at any arbitrary frequency. Eq. (16) may be rewritten in a somewhat simpler form as follows:

$$\eta(\omega) = \frac{\beta}{\dfrac{1}{{\omega_N}^2} + (\beta+1)^2} \qquad (22)$$

The task is to find the value of β that that maximizes the efficiency function Eq. (22) at any arbitrary frequency ω_N. As before, this my be done by calculating the partial derivative of η with respect to β, equating to zero, and solving for the β. This value of beta can then be substituted back into Eq. (22) and then an expression for maximum efficiency, as a function of ω_N, can be derived. When this is done (a fair amount of work!), the following value of β results in maximum efficiency at an arbitrary ω_N:

$$\beta = \sqrt{1 + \frac{1}{{\omega_N}^2}} \qquad (23)$$

When this value of β is substituted in Eq. (22), the following relationship for maximum efficiency is obtained (also a fair amount of work!):

$$\eta_{\max} = \frac{1}{2\left(1 + \sqrt{1 + \dfrac{1}{{\omega_N}^2}}\right)} \qquad (24)$$

In various frequency regions, this maximum efficiency function takes on the following values:

for $\omega_N >> 1$

(high frequencies)

$$\eta_{\max} = \frac{1}{4} \qquad (25\%) \qquad (25)$$

for $\omega_N = 1$

(corner frequency)

$$\eta_{\max} = \frac{1}{2(1+\sqrt{2})} \approx 0.207 \qquad (20.7\%)$$

, and (26)

for $\omega_N << 1$

(low frequencies)

$$\eta_{\max}(\omega_N) = \frac{\omega_N}{2}$$

, rolls off at 3-dB per octave. (27)

To relate the maximum efficiency function Eq. (24) to values of *ka*, Eq. (10) can be used to rewrite Eq. (24) as a function of *ka* as follows:

$$\eta_{\max}(ka) = \frac{1}{2\left(1+\sqrt{1+\left(\frac{16}{3\pi}\bullet\frac{1}{ka}\right)^2}\right)} \approx \frac{1}{2\left(1+\sqrt{1+\left(\frac{1.7}{ka}\right)^2}\right)}.$$

(28)

Eq. (28), in turn, can be approximated in various frequency regions, as follows:

for $ka >> 1$

(high frequencies)

$$\eta_{\max} = \frac{1}{4} \qquad (25\%)$$

, (29)

for $ka = 1$

$$\eta_{\max}(1) = \frac{1}{2\left(1+\sqrt{1+\left(\frac{16}{3\pi}\right)^2}\right)} \approx 0.168 \qquad (16.8\%)$$

, and (30)

for $ka << 1$

(low frequencies)

$$\eta_{\max}(ka) = \frac{3\pi}{32} ka \approx 0.29\, ka$$

, rolls off at 3-dB per octave. (31)

A plot of Eq. (28), using the approximate analogous load circuit components of Eqs. (6) and (7), is shown in Fig. 10. A plot of maximum efficiency using the actual load values (not shown), rather than the approximate analogous load values, reveals that the two curves are essentially equal, except in the range of $1 < ka < 3$, where the value differs at most by about 0.4 dB. Fig. 10 shows the maximum efficiency of the radiator as a function of ka. It can be thought of as the maximum possible efficiency of the radiator at any arbitrary frequency.

As will be shown later, this is not an absolute upper limit of efficiency, but is a reasonable upper limit of efficiency for most real-world realizable designs. Some unrealizable designs, with very low moving masses ($\psi < 1$), that use resonance to boost the efficiency in a narrow frequency range near cutoff, can exceed this limit.

Eq. (31) can be rewritten in terms of the natural frequency variable f, as follows:

$$\text{for } f << \frac{c}{2\pi a} \approx \frac{54.6}{a}$$

(low frequencies)

$$\eta_{\max}(f) = \frac{3\pi^2}{16}\frac{fa}{c} \approx \frac{fa}{185}, \text{ for } f \text{ in Hz and } a \text{ in meters.} \quad (32)$$

Eq. (31) can be also be rewritten in terms of wavelength λ and effective piston diameter d, as follows:

$$\text{for } ka < 0.3$$

$$\eta_{\max}(\lambda, d) = \frac{3\pi}{32}ka = \frac{3\pi}{32}\frac{2\pi a}{\lambda} = \frac{3\pi^2}{32}\frac{2a}{\lambda} = \frac{3\pi^2}{32}\frac{d}{\lambda} \approx 0.93\frac{d}{\lambda}. \quad (33)$$

Eq. (33) can be further rounded off to approximately (within about 8%):

$$\text{for } \frac{d}{\lambda} < 0.1$$

$$\eta_{\max} \approx \frac{d}{\lambda}. \quad (34)$$

This last surprisingly simple result relates the maximum efficiency to the ratio of the effective diaphragm diameter of the radiator, to the wavelength of the radiated sound. For example, at 40 Hz where the wavelength is about 338 inches (8.6 m), a 12-in advertised diameter driver having an effective diameter of 9.5 inches (0.24 m), could have at most an efficiency of about $9.5 / 338 \approx 0.028 = 2.8\%$ (exact value of 2.46% using Eq. (28), using the rule of thumb that assumes that the driver radius in cm is the same as the advertised diameter in inches). Remember that this would only be for a driver whose moving mass was significantly less than its air-load mass; a very challenging driver design!

It is interesting to compare relationship (34) to that reported by Small for the midband reference efficiency of a direct-radiator loudspeaker system [11, Eq. (1)]:

$$\eta_0 = 4\pi^2 K_\eta \left[\frac{V_B}{\lambda_3{}^3} \right] \tag{35}$$

where

η_0	asymptotic efficiency for radiation into half-space ,
K_η	complete efficiency constant, or figure of merit,
V_B	net internal volume of the enclosure, and
λ_3	wavelength corresponding to the system half-power or -3 dB cutoff frequency f_3

This can be rewritten in terms of a side dimension D for a cube of volume V_B ($= D^3$) as follows:

$$\eta_0 = 4\pi^2 K_\eta \left[\frac{D}{\lambda_3} \right]^3 \tag{36}$$

This form relates the reference efficiency to the cube of the ratio of the linear dimension of the radiator to the wavelength of sound. As Small states in this same reference [11]:

> "In common with many other transducers and radiators, it is the small dimensions of the system in relation to wavelength that restrict the radiation efficiency to a very low value; the magnitude of the bracketed factor [*in Eq. (35)*] for typical domestic high-fidelity sound reproducers is restricted to the order of 10^{-4}."

It is informative to compare Eqs. (34) and (36), both show how efficiency depends on the ratio of the linear dimensions of the radiator to the wavelength of the radiated sound. The maximum efficiency relationship in Eq. (34) depends only on the first power of the ratio (x2 or +3 dB for each doubling of value) whereas, Eq. (36) shows a very strong third power relationship (x8 or +9 dB for each doubling of the value). Note that both bracketed values are ratios of linear dimensions to wavelength, but in the first case it is the ratio between the effective diameter (d) of the radiator to the wavelength of radiated sound, and in the second case it is the ratio between the largest overall dimension of the complete loudspeaker enclosure (D) to the radiated wavelength at the system's cutoff frequency.

4.2.6. Maximum Efficiency vs Driver Diameter

Fig. 11, generated using Eq. (28), shows the maximum efficiency for various advertised driver diameters as a function of frequency. The maximum efficiency for drivers having advertised diameters of 2, 4, 8, 10, 12, 15, 18, 24, and 30 inches are shown. Using the previous example of a 12-in driver at 40 Hz, having a maximum efficiency of roughly 2.5%, you won't be able to get the efficiency above this value at 40 Hz no matter high large you make its enclosure or manipulate its parameters! The graph emphasizes the point that if you want high efficiency at low frequencies, you need a large diameter radiator or equivalently an array of smaller diameter radiators having the same combined area of the larger radiator.

4.3 Efficiency With Infinite Suspension Compliance and Finite Moving Mass

This situation differs from the previous analyzed situation in that the moving mass of the driver diaphragm assembly is included in the model. The main effect of adding the moving mass to the model is a limitation of the high-frequency response. The response will roll off at 6-dB-per-octave above the frequency where the reflected reactance of the moving mass equals the voice coil resistance ($\omega_1 = B^2l^2/(R_E M_{MD})$). The addition of the moving mass to the previous model changes the response function into a second-order bandpass filter with significant frequencies ω_1 and ω_2. The Q and bandwidth of the resultant second-order bandpass filter depend on the relative locations of ω_1 and ω_2. For real-world designs, ω_1 is always very much less than ω_2, thus making the efficiency low. This constraint also reverses the rolls of ω_1 and ω_2 and makes the moving mass corner frequency ω_1 coincide with the low-frequency rolloff, and the radiation reactance - resistance corner ω_1 coincide with the high-frequency rolloff.

4.3.1. Model

The model for the situation is the same as the last model, except for the addition of a capacitor C_1, from the junction of R_1 and C_2 to ground, corresponding to the moving mass of the driver diaphragm. The circuit model corresponding to this situation is shown in Fig. 12. The Bl product acts as a scale factor to change the relative impedance level of the moving mass component as compared to the resistance of the voice coil (R_E).

4.3.2. Response Function

If the approximate analogous circuit is used for the radiation load, the response function for this situation is a pure second-order band-pass filter with the following response function:

$$G(s) = \frac{\dfrac{1}{R_1 C_1} s}{s^2 + \left[\dfrac{1 + \dfrac{C_2}{C_1}}{R_2 C_2} + \dfrac{1}{R_1 C_1} \right] s + \dfrac{1}{R_1 C_1 R_2 C_2}} \quad (37)$$

Eq. (37) may be converted to magnitude squared form:

$$|G(j\omega)|^2 = \frac{\left(\dfrac{1}{R_1{}^2 C_1{}^2}\right)\omega^2}{\left(\dfrac{1}{R_1C_1R_2C_2} - \omega^2\right)^2 + \left(\dfrac{1+\dfrac{C_2}{C_1}}{R_2C_2} + \dfrac{1}{R_1C_1}\right)^2 \omega^2} \tag{38}$$

4.3.3. Center Frequency and Q

Eq. (37) may be rewritten using the variables: $\psi = C_1 / C_2$, $\omega_1 = 1 / (R_1C_1)$, and $\omega_2 = 1 / (R_2C_2)$:

$$G(s) = \frac{\omega_1 s}{s^2 + \left[\left(1+\dfrac{1}{\psi}\right)\omega_2 + \omega_1\right]s + \omega_1\omega_2} \tag{39}$$

This equation can be equated to the standard form for a second-order bandpass filter:

$$G(s) = \frac{As}{s^2 + \dfrac{\omega_0}{Q}s + \omega_0{}^2}, \tag{40}$$

and in turn be solved for the center frequency ω_0 and Q. These are given as follows:

$$\omega_0 = \sqrt{\omega_1\omega_2}\text{, and} \tag{41}$$

$$Q = \frac{\sqrt{\dfrac{\omega_1}{\omega_2}}}{1+\dfrac{1}{\psi}+\dfrac{\omega_1}{\omega_2}}. \tag{42}$$

If you assume that $\omega_1 << \omega_2$ (moving mass corner frequency much lower than the frequency where $ka = 1$), and $\psi >> 1$ (moving mass much larger than air-load mass, a typical direct radiator), then Eq. (42) simplifies to:

$$Q \approx \sqrt{\frac{\omega_1}{\omega_2}} \,. \tag{43}$$

This equation clearly shows that if $\omega_1 << \omega_2$ then Q will also be low and thus the bandwidth wide. Also note from Eq. (42) that if $\psi << 1$ (moving mass much <u>smaller</u> than air-load mass, a not-so realizable design!), then the Q is also low and bandwidth wide.

4.3.4. Efficiency Function and Relationships

Eq. (38) can be substituted into Eq. (11) to yield the efficiency function:

$$\eta(\omega) = |G(j\omega)|^2 \frac{R_1}{R_2} = \frac{\left(\dfrac{1}{R_1 R_2 C_1^{\,2}}\right)\omega^2}{\left(\dfrac{1}{R_1 C_1 R_2 C_2} - \omega^2\right)^2 + \left(\dfrac{1+\dfrac{C_2}{C_1}}{R_2 C_2} + \dfrac{1}{R_1 C_1}\right)^2 \omega^2} \,. \tag{44}$$

The efficiency at the peak of this function occurs at the center of the second-order band-pass function Eq. (37), at center frequency $\omega = \omega_0 = 1 / (R_1 C_1 R_2 C_2)$. When Eq. (44) is evaluated at this frequency, the following efficiency is obtained after some manipulation:

$$\eta(\omega = \omega_0) = \eta_{peak} = \frac{\dfrac{R_1}{R_2}}{\left[1 + \dfrac{R_1}{R_2}\left(1 + \dfrac{C_1}{C_2}\right)\right]^2} \,. \tag{45}$$

Eq. (45) may be rewritten using the variables: $\psi = C_1 / C_2$, and $\beta = R_1 / R_2$ resulting in:

$$\eta_{peak} = \frac{\beta}{\left[1 + \beta\,(1 + \psi)\right]^2} \,. \tag{46}$$

This rather simple relationship shows how the peak efficiency depends on the moving mass to air-load mass ratio ψ, and the ratio β between the voice-coil resistance and the reflected real part of the radiation load. Fig. 13 (a) shows a two-dimensional plot of Eq. (46) plotted against ψ and β. Also shown in Fig. 13 (b) is a plot of Eq. (46) but multiplied by the factor $1+\psi$, to show the trajectory of the maximum in the $\psi\beta$ plane. The reason for the multiplication will be evident in the next section.

4.3.5. Efficiency vs Moving Mass to Air-Load Mass Ratio

Eq. (46) can be maximized with respect to β, to yield a maximum efficiency relationship which is a function of ψ. When this is done, the following value of β is found to maximize Eq. (46):

$$\beta = \frac{1}{1+\psi}. \tag{47}$$

When this value is substituted in Eq. (46) the following equation for maximum efficiency results:

$$\eta_{\max} = \frac{1}{4(1+\psi)} = \frac{0.25}{1+\psi}. \tag{48}$$

Fig. 14 shows a plot of this relationship. The plot starts at an efficiency of 25% and then at $\psi = 1$ starts to smoothly decrease 3 dB for each doubling of ψ. This equation again emphasizes that high efficiencies are only attained when the moving mass to air-load mass ratio is low. If you assume that the moving mass is much larger than the air load mass ($\psi >> 1$), then Eq. (48) simplifies to:

$$\eta_{\max} = \frac{1}{4\psi} = \frac{0.25}{\psi}. \tag{49}$$

If the drivers mechanical parameters for ψ are substituted into Eq. (49), the following results:

$$\eta_{\max} = \frac{2\rho_0 a^3}{3M_{MD}}. \tag{50}$$

In a typical driver or array of drivers, the moving mass usually increases in direct proportion to cone area or radius squared (a^2), but the air load mass increases in direct proportion to radius cubed (a^3), which makes the maximum efficiency rise in direct proportion to the the driver's radius (or linear dimension). This means that as the driver size increases or as the number of like speakers in an array increases, the maximum possible efficiency also increases.

This behavior can be contrasted with the Thiele/Small reference efficiency equation (reproduced here later as Eq. (54)), where the efficiency increases in direct proportion to the forth power of the radius (a^4 or area squared), and decreases in direct proportion to moving mass squared (a^4 again). This predicts that efficiency is essentially independent of cone size.

5. COMPARISON WITH THIELE-SMALL EFFICIENCY

The equation for peak efficiency Eq. (46), derived here, can be simplified with with certain assumptions that results in an equation that yields Small's reference efficiency. Two versions of Small's reference efficiency are derived, one which is a function of the total mass, including both the moving and air-load mass, and the second which depends on the moving mass only. These two efficiency equations are then used to derive error relationships which show the dB error between the true efficiency and the Thiele-Small reference efficiency.

5.1 Derive Thiele-Small Efficiency from Maximum Efficiency Relationships

Eq. (46) may be further simplified by assuming that $\beta\,(1+\psi) >> 1$ which is equivalent to $\beta >> 1/(1+\psi)$ or $\psi >> (1/\beta - 1)$:

$$\eta_{peak} = \frac{\beta}{\left[1+\beta\,(1+\psi)\right]^2} \approx \frac{\beta}{\left[\beta\,(1+\psi)\right]^2} = \frac{1}{\beta\,(1+\psi)^2} . \tag{51}$$

The assumptions that created this equation are the same as Thiele and Small used to simplify their analysis of the direct radiator model [1] [2]. If β and ψ are expanded into their equivalent physical parameter values the following results:

$$\psi = \frac{C_1}{C_2} = \frac{C_{MES}}{C_{EAR}} = \frac{\dfrac{M_{MD}}{B^2 l^2}}{\dfrac{M_{MA}}{B^2 l^2}} = \frac{M_{MD}}{M_{MA}} = \frac{M_{MD}}{\frac{8}{3}\rho_0 a^3} = \frac{3}{8\rho_0} \bullet \frac{M_{MD}}{a^3}$$

$$\approx 0.31 \bullet \frac{M_{MD}}{a^3} \quad \text{, and} \tag{52}$$

$$\beta = \frac{R_1}{R_2} = \frac{R_E}{R_{EAR}} = \frac{R_E}{\dfrac{9\pi^2}{128} \bullet \dfrac{B^2 l^2}{\rho_0 c S_D}} = \frac{128\rho_0 c}{9\pi^2} \bullet \frac{S_D R_E}{B^2 l^2}$$

$$\approx 1879 \frac{S_D R_E}{B^2 l^2} \tag{53}$$

If Eqs. (52) and (53) are substituted into Eq. 49, the following results:

$$\eta_{peak} = \frac{1}{\beta\,(1+\lambda)^2} = \frac{1}{\dfrac{128\rho_0 c}{9\pi^2} \bullet \dfrac{S_D R_E}{B^2 l^2}\left(1+\dfrac{M_{MD}}{M_{AR}}\right)^2}$$

$$= \frac{9\pi^2 B^2 l^2}{128\rho_0 c\, S_D R_E \dfrac{(M_{AR}+M_{MD})^2}{M_{AR}{}^2}} = \frac{9\pi^2 B^2 l^2 M_{AR}{}^2}{128\rho_0 c\, S_D R_E M_{MS}{}^2}$$

$$= \frac{9\pi^2 B^2 l^2 \left(\dfrac{8}{3}\rho_0 a^3\right)^2}{128\rho_0 c\, \pi a^2 R_E M_{MS}{}^2} = \frac{9(64)\rho_0}{128(9)\pi c} \bullet \frac{B^2 l^2 \pi^2 a^4}{R_E M_{MS}{}^2}$$

$$= \frac{\rho_0}{2\pi c} \bullet \frac{B^2 l^2 S_D{}^2}{R_E M_{MS}{}^2}. \qquad (54)$$

Eq. (54) is recognized as being Small's reference efficiency η_0 [2, Eq. (31), p. 391], where M_{MS} includes the moving mass *and* the air-load mass.

Eq. (51) can be simplified further by assuming that the moving mass M_{MD} is much larger than the air load mass M_{AR}, ie $\psi >> 1$:

$$\eta_{peak} \approx \frac{1}{\beta\,(1+\psi)^2} \approx \frac{1}{\beta\,\psi^2}. \qquad (55)$$

In like manner, the physical equivalents of β and ψ can be substituted into Eq. (55) and the following results:

$$\eta_{peak} \approx \frac{\rho_0}{2\pi c} \bullet \frac{B^2 l^2 S_D{}^2}{R_E M_{MD}{}^2}. \qquad (56)$$

This is recognized as being Small's reference efficiency but using the moving mass only, *without* the air-load mass being added in. I shall call this efficiency η_0', the prime denoting that the reference efficiency is computed using the moving mass only.

5.2 Efficiency Error Relationship Including Radiation Air Mass Load

Eq. (51) can be solved for β and this value substituted into Eq. (46) to yield a relationship between the actual efficiency η_{peak}, and the Thiele-Small efficiency η_0. The value of β from Eq. (51) is:

$$\beta = \frac{1}{\eta_{peak}(1+\psi)^2} = \frac{1}{\eta_0(1+\psi)^2} \quad (57)$$

This value can be substituted in Eq. (46) yielding:

$$\eta_{peak} = \frac{\eta_0}{[\eta_0(1+\psi)+1]^2} \quad (58)$$

The ratio between η_0 and η_{peak} can be formed and equated to an error value E as follows:

$$\frac{\eta_0}{\eta_{peak}} = [\eta_0(1+\psi)+1]^2 = E \quad (59)$$

Note that the calculated Thiele/Small reference efficiency η_0 will *always* be larger than the actual efficiency η_{peak}, thus making the error value greater than one ($E > 1$).

For the error to be between unity (no error) and E, this ratio must be equal to or less than E:

$$[\eta_0(1+\psi)+1]^2 \le E \quad (60)$$

This inequality can in turn be solved for η_0 as follows:

$$\eta_0 \le \frac{1}{1+\psi}\left(\sqrt{E}-1\right) \quad (61)$$

Eq. (61) indicates the range of η_0 so that the error will be within unity (no error) and error E. Fig. 15 shows constant error contours of this relationship for error values of 1 dB ($E = 10^{+0.1} \approx 1.26$), 3 dB ($E = 10^{+0.3} \approx 2.0$), and 6 dB ($E = 10^{+0.6} \approx 4.0$). For each contour and at a specific value of ψ, η_0 must be equal to or less than the indicated value. Note that as ψ increases, the allowed range of efficiencies decreases.

5.3 Efficiency Error Relationship Not Including Radiation Air Mass Load

A similar error analysis for η_0', the efficiency calculated using the moving mass only, yields the following inequality:

$$\eta_0' \le \frac{1}{\psi}\left(\sqrt{E} - 1 - \frac{1}{\psi}\right). \qquad (62)$$

Fig. 16 shows constant error contours of Eq. (62) for the same error values of Fig. 15. Note the added regions on the left of the graph, to the left of where the constant error contours dive to zero, which severely restrict the range of valid efficiencies. This indicates that if the air-load mass is not added to the moving mass, the efficiency error can be extremely large if the moving mass to air-load mass ratio ψ is small.

6 Bl PRODUCT THAT MAXIMIZES EFFICIENCY

A specific value of Bl can be selected that maximizes the power available efficiency. To derive this value, Eq. (47) can be recast into a form that uses the driver's mechanical parameters and then solved for Bl. This value of Bl will then maximize the drivers power available efficiency for the particular combination of diaphragm radius a, moving mass M_{MD}, and dc resistance R_E. When the mechanical parameter equivalents of ψ and β, Eqs. (52) and (53), are substituted into Eq. (47), and then solved for Bl, the following results:

$$\begin{aligned} Bl &= \sqrt{\frac{128\rho_0 c}{9\pi}}\; a\; \sqrt{R_E\left(\frac{3}{8\rho_0}\bullet\frac{M_{MD}}{a^3}+1\right)} \\ &\approx 43.3\; a\; \sqrt{R_E\left(0.31\frac{M_{MD}}{a^3}+1\right)}, \end{aligned} \qquad (63)$$

where

- a effective radius of driver diaphragm
- M_{MD} mechanical moving mass of driver diaphragm assembly *not* including air load
- R_E dc resistance of driver voice coil.

Eq. (63) may be further simplified by assuming that the moving mass is much larger than the air load mass ($\psi >> 1$, the left term in the braces is much larger than one):

$$\begin{aligned} Bl &\approx \sqrt{\frac{16c}{3\pi}}\;\sqrt{\frac{R_E M_{MD}}{a}} \\ &\approx 24.1\;\sqrt{\frac{R_E M_{MD}}{a}}. \end{aligned} \qquad (64)$$

7. EXAMPLE

In this section, I use one of Small's example driver designs [3] to illustrate the efficiency relationships developed in this paper.

7.1 R. E. Small Example

Small calculates the following parameter specifications for a 12-inch advertised-diameter driver to be used in a closed-box air-suspension loudspeaker system having a Butterworth second-order high-pass *B2* response at 40 Hz, in a 2 ft^2 enclosure, with a compliance ratio of 5:

f_s = 16.3 Hz
Q_{ES} = 0.336
V_{AS} = 0.283 m^3 (10 ft^3)
η_0 = 0.35%
a = 0.12 m (for a 12-inch driver)
S_D = 4.5 x 10^{-2} m^2
C_{MS} = 9.9 x 10^{-4} m/N
M_{MS} = 97 g
M_{MA} = $8\rho_0 a^3 / 3 \approx 3.23\, a^3$ = 5.5 g (assuming front air load of infinite baffle only, Small included another 4.5 g for the enclosure)
M_{MD} = $M_{MS} - M_{MA}$ = 91.5 g
B^2l^2/R_E = 30 N ∘ s/m. (electro-magnetic damping factor)
R_E = 6.5 ohms (typical for an 8 ohm impedance rating)
Bl = 14 T ∘ m

where

f_s resonance frequency of unenclosed driver
Q_{ES} Q of driver at f_s considering electrical resistance R_E only
V_{AS} volume of air having same acoustic compliance as driver suspension
η_0 Thiele/Small reference efficiency.
a effective radius of driver diaphragm
S_D effective surface area of driver diaphragm (= πa^2)
C_{MS} mechanical compliance of driver suspension
M_{MS} mechanical moving mass of driver diaphragm assembly including air load (= $M_{MD} + M_{MA}$)
M_{MA} mechanical air-load mass on driver diaphragm
M_{MD} mechanical moving mass of driver diaphragm assembly not including air load
B^2l^2/R_E electro-magnetic damping factor
R_E dc resistance of driver voice coil
Bl product of magnetic flux density in driver air gap and length of voice-coil conductor in magnetic gap

Neglecting driver suspension compliance and box compliance, the component values for the equivalent electric network of Fig. 12 are:

$R_1 = R_E = 6.5$ Ohms

$C_1 = M_{MD} / B^2l^2 = 0.0915 / 14^2 \approx 4.67 \times 10^{-4}$ F = 467 μF

$R_2 = B^2l^2 / 1879a^2 = 14^2 / 1879\ (0.12^2) \approx 7.2$ Ohms

$C_2 = M_{MA} / B^2l^2 = 0.0055 / 14^2 \approx 2.8 \times 10^{-5}$ F = 28 μF

Other parameters, derived in this paper, result in the following values:

$\beta = R_1 / R_2 = 6.5 / 7.2 \approx 0.90$

$\psi = C_1 / C_2 = 467 / 28 \approx 16.7$

$\omega_1 = 1 / R_1C_1 \approx 329$ radians per sec

$f_1 = \omega_1 / 2\pi \approx 52.4$ Hz

$\omega_2 = 1 / R_2C_2 \approx 4960$ radians per sec

$f_2 = \omega_2 / 2\pi \approx 789$ Hz ($ka = 1$ at 455 Hz)

From Eq. (48), the maximum efficiency for this value of moving mass to air load mass ratio is:

$$\eta_{max} = 0.25 / (1 + \psi) = 0.25 / (1 + 16.7) = 0.0141 \approx 1.4\ \%.$$

Note that the maximum efficiency (1.4 %) is *higher* than the designed reference (0.35%), as it should be. The value is higher by a factor of 4 or 6 dB. For the chosen cone size and moving mass, the value of maximum efficiency represents a limit that cannot be exceeded no matter how you manipulate the driver's remaining parameters.

The actual efficiency for this combination of parameters is given by the peak efficiency function Eq. (58) and yields the following:

$$\eta_{peak} = \frac{\beta}{[1+\beta\,(1+\psi)]^2} = \frac{0.9}{[1+0.9(1+16.7)]^2} \approx 0.00314 = 0.314\%$$

Note that this value (0.314 %) is *below* the designed reference efficiency (0.35%) by a factor of 0.897 (about -0.5 dB), as it also should be. This is because the traditional design method neglects the radiation air load in the equivalent circuit model (but does lump in the air-load mass into the total moving mass). Note that this design ($\eta_0 = 0.35$ % and $\psi = 16.7$) appears somewhat below the 1 dB error contour line in Fig. 15, as expected.

The maximum efficiency for this driver design is attained when the Bl product is set to the value given by Eq. (63):

$$Bl = 43.3a\sqrt{R_E\left(0.31\frac{M_{MD}}{a^3}+1\right)}$$

$$= 43.3(0.12)\sqrt{6.5\left(0.31\frac{0.0915}{0.12^3}+1\right)} \approx 55.3\ T\cdot m$$

This value is approximately $55.3 / 14 \approx 3.95 \approx 4$ times larger than the original Bl product.

Fig. 17 shows various efficiency frequency responses corresponding to the different conditions in this example. The original designed high-pass response is shown in (a), (b) shows the actual response predicted in this paper including the actual air load (with correct driver and box compliance), while (c) shows the actual efficiency response but with infinite driver and box compliance and (d) shows the response for the maximum efficiency condition when the *Bl* product is set to the proper value. Also shown (e) is the maximum efficiency curve for this particular driver size using Eq. (28). Observe that curves (b), (c), and (d) are band-pass responses, while (a) and (d) are high-pass responses. Note that target response, using the Thiele/Small approximations, and the actual response are very close together below 500 Hz. Above 500 Hz the actual response rolls off at 6-dB per octave while the target response continues at the 0.35% plateau level. Note also that the maximum efficiency alignment ($Bl = 55.3$ T•m) has a very narrow response range.

8. SUMMARY

The widely used methods of direct-radiator loudspeaker system analysis, based on the pioneering work of Thiele and Small, neglect the radiation impedance components in deriving the system response functions. Neglecting these components greatly simplifies the analysis and design procedure. This study provides a systematic analysis of the direct-radiator model with the radiation impedance components included. It was found that the absolute maximum efficiency is 25%, because of the commonly used definition of direct-radiator efficiency, which defines a fictitious input power that is developed in the fixed dc resistance of the driver's voice coil. The assumed fixed input resistance simplifies the analysis, because the input power is then independent of frequency.

In this study, I derived an equation yielding the absolute maximum direct-radiator efficiency as a function of frequency and radiator size. Above $ka = 1$, the maximum efficiency attains a plateau of 25%, below $ka = 1$, the maximum efficiency decreases in proportion to ka. For low frequencies, $ka < 0.3$, maximum efficiency is approximately equal to the effective diameter of the radiator divided by the wavelength of the radiated sound. This means that a typical 12-in advertised diameter driver can have an efficiency no more than about 2.5%, no matter how you manipulate its parameters.

If finite moving mass is included in the analyzed model, the system efficiency vs frequency turns into a second-order (single-tuned) bandpass response. The efficiency at the peak of this bandpass response depends only on the on the moving mass to air-load mass ratio ψ, and the ratio β between the voice-coil resistance and the reflected real part of the radiation load. If β is chosen appropriately, the maximum efficiency at the peak of the bandpass is found to depend only on ψ. For $\psi << 1$ (a very challenging driver design) efficiency approaches 25%, for $\psi >> 1$ (high moving mass in relation to air-load mass) maximum efficiency decreases in direct proportion to increasing ψ. This relationship emphasizes that high efficiencies are only attained when the moving mass to air-load mass ratio is low. For a typical driver whose moving mass is 20 times its air-load mass, its efficiency can be no higher than 1.2%.

Relationships were derived that show the efficiency error in using the Thiele/Small direct-radiator driver reference efficiency equation. The Thiele/Small reference efficiency over estimates the efficiency, and is *always* larger than the actual efficiency. For a typical driver with a moving mass to air-load mass ratio in the range of 10 to 20, the Thiele/Small efficiency must be less than about 0.6%, to limit over-estimation errors to 1 dB or less. If the air-load mass is not added to the total moving mass of the driver, in calculating the Thiele/Small efficiency, errors can be much greater.

A formula yielding the driver's *Bl* product that maximizes driver efficiency was derived. Unfortunately, this value of *Bl* in addition to maximizing efficiency, often changes the response shape into an unacceptably narrow higher-center-frequency bandpass.

A typical driver synthesis example of Small's was analyzed to determine efficiency errors and response shapes.

9. ACKNOWLEDGEMENT

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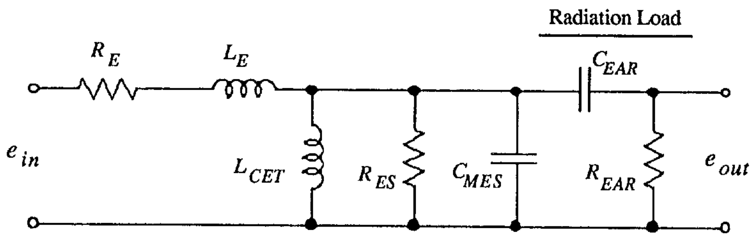

Fig. 1. Relatively complete mobility-type analogous circuit model (voltage <-> velocity or volume velocity, current <-> force or pressure) of a direct-radiator loudspeaker mounted in a closed-box enclosure. Inductor L_{CET} models the total system compliance, which includes driver suspension compliance and box rear air-cavity compliance. Capacitor C_{MES} models the total moving mass of the driver less radiation air mass. The values of R_{EAR} and C_{EAR}, which model the real and imaginary parts of the driver's radiation load, are not constant and vary with frequency. For low-frequencies ($ka < 0.3$), the approximate constant values can be used, Eqs. (6) - (7).

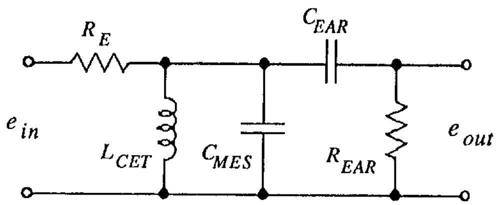

Fig. 2. Simplified analogous circuit model of a direct-radiator loudspeaker mounted in a closed-box enclosure. The components modeling driver voice-coil inductance (L_{CET}) and driver mechanical losses (R_{ES}) have been removed.

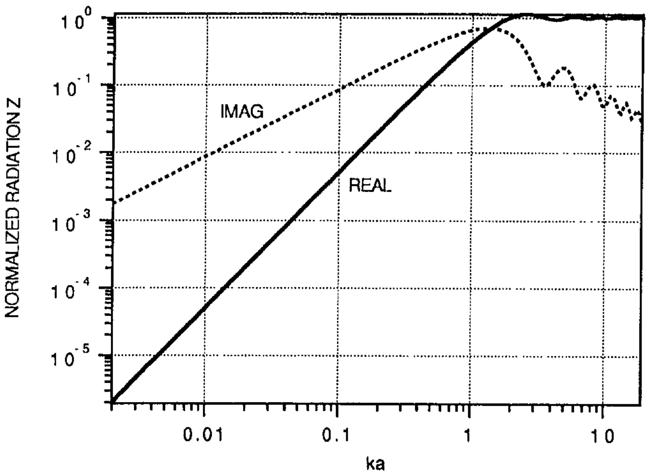

Fig. 3. Real and imaginary parts of the normalized radiation impedance of the air load upon one side of a plane circular piston mounted in an infinite flat baffle (2π steradian half-space acoustic load). The impedance-type analogous circuit, that corresponds to this radiation impedance, is a parallel RL circuit whose component values vary with frequency.

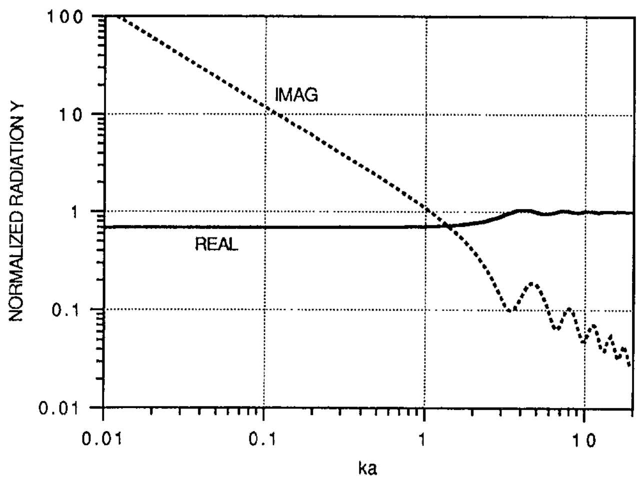

Fig. 4. Real and imaginary parts of the normalized radiation mobility of the air load upon one side of a plane circular piston mounted in an infinite flat baffle (2π steradian half-space acoustic load). This is just the reciprocal of the radiation impedance function of Fig.3. The mobility-type analogous circuit, that corresponds to this radiation mobility, is a parallel RC circuit whose component values vary with frequency.

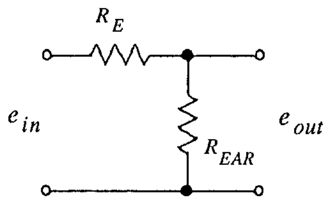

Fig. 5. Analogous circuit of Fig. 2 with infinite suspension compliance, zero moving mass and infinite air-load mass. This situation corresponds to the condition where all the components that might impede the flow of power from source to load, have been removed from the circuit model. Only the resistance of the voice coil and the real part of the radiation load, remain.This model is the simplest possible case and just includes the driver dc voice coil resistance ($R_E = R_1$) connected directly to the radiation load ($R_{EAR} = R_2$).

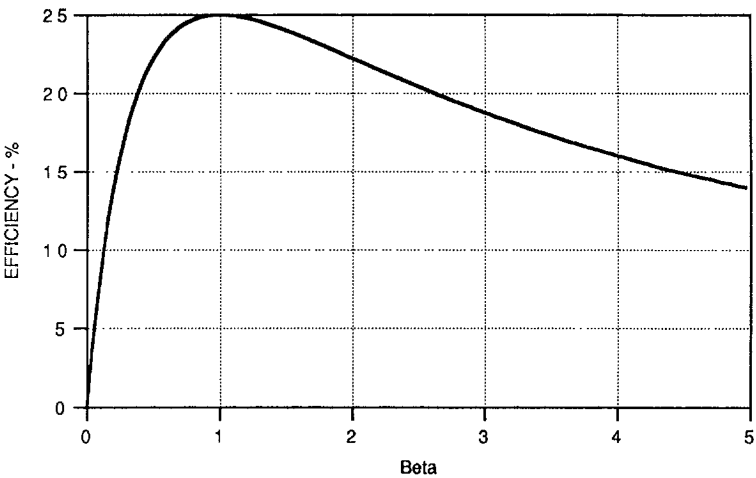

Fig. 6. Plot of efficiency (Eq. 13) vsβ (= R_E / R_{EAR} or R_1/R_2) for the analogous circuit of Fig. 5. The 25% peak efficiency occurs when the reflected radiation load resistance equals the voice-coil resistance ($\beta = 1$, or $R_E = R_{EAR}$).

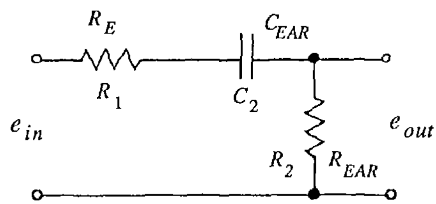

Fig. 7. Analogous circuit of Fig. 2 with infinite suspension compliance and zero moving mass. *This situation differs from Fig. 5 in that the radiation air mass load is included in the model.* This corresponds to the situation of the voice-coil resistance directly driving the complete radiation load, including both real and imaginary parts. This is the circuit used to calculate the maximum possible efficiency of the direct-radiator loudspeaker.

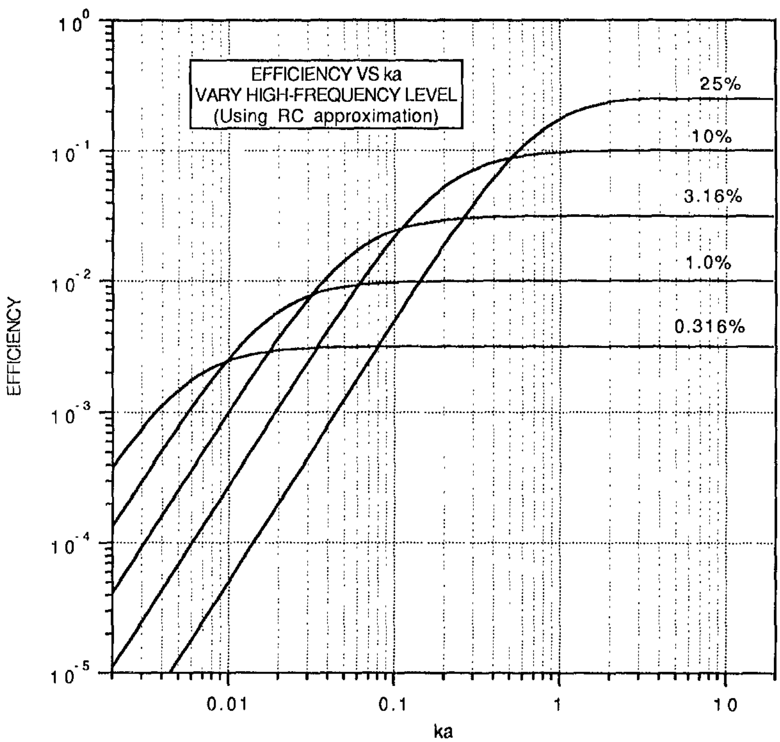

Fig. 8. Plots of efficiency vs *ka* for various values of β (= R_E/R_{EAR}), for the analogous circuit of Fig. 7, using Eq. (17). The approximately constant values of the radiation load components, R_{EAR} and C_{EAR} (Eqs. (6) and (7)), were used. Note, that as the plateau efficiency decreases, the low-frequency cutoff extends lower in frequency. The value of β was varied ($\beta \geq 1$) to make the high-frequency plateau efficiency equal to the following specific values:

η, %	β
25.0	1.0
10.0	7.87
3.16	29.6
1.0	98.0
0.316	313.6

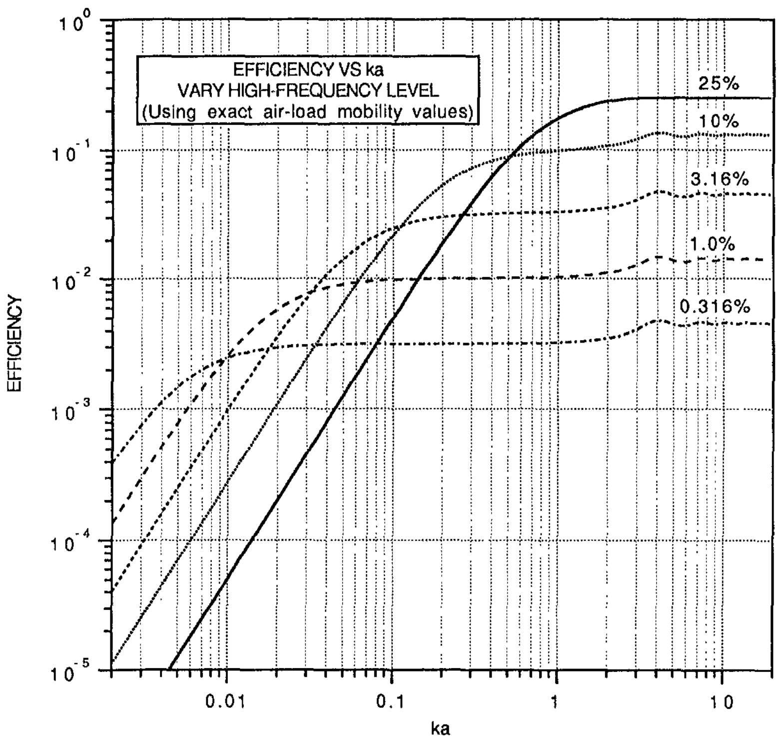

Fig. 9. Repeat of Fig. 8, but using the actual radiation air-load mobility values of Fig. 4, which vary with frequency, for the radiation load components R_{EAR} and C_{EAR}. In this case, β was assigned values that made the efficiency plateau values, for the $ka < 1$ frequency range, equal to the chosen efficiencies in Fig. 8. The main effect of using the actual load values, is an increase of efficiency by about 20% and added ripple above $ka = 2$.

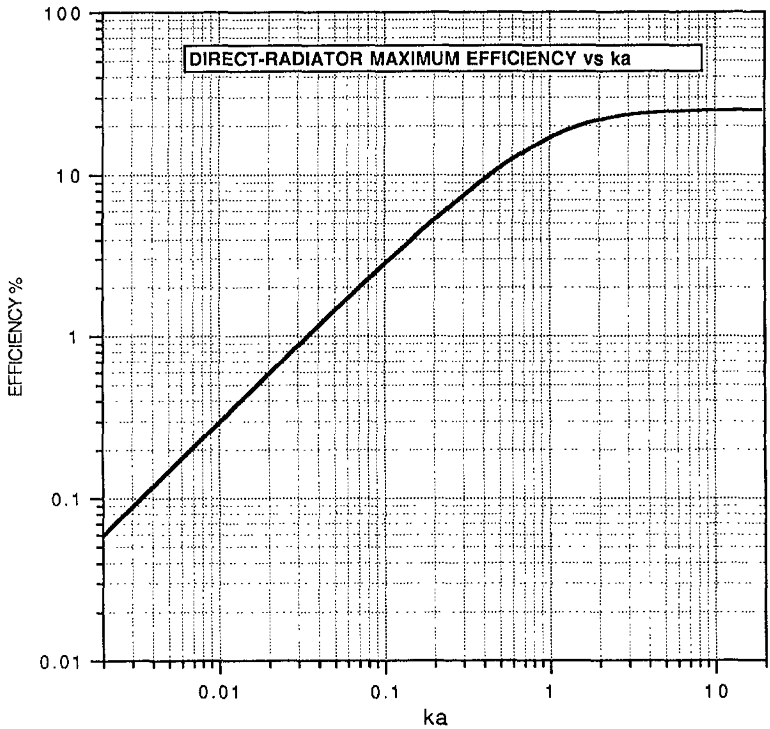

Fig. 10. Plot of direct-radiator maximum efficiency vs *ka* using Eq. (28), which is based on using the approximate analogous load circuit components of Eqs. (6) and (7). Note that this curve is the locus of the half-power points of the curves in Figs. 8 and 9. The curve can be thought of as the maximum possible efficiency of a radiator at any arbitrary frequency.

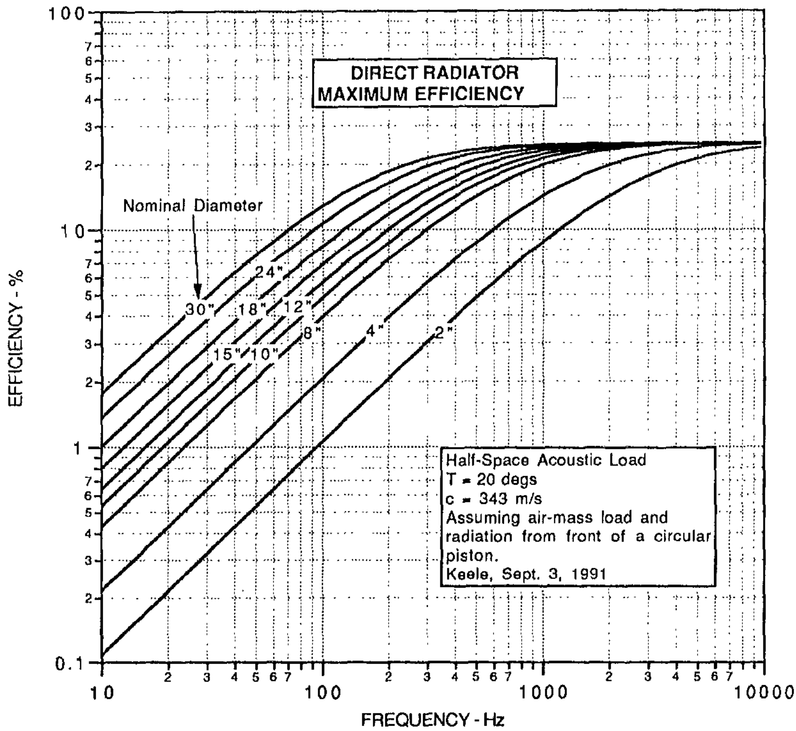

Fig. 11. Theoretical maximum efficiency of a direct-radiator driver vs frequency, for various nominal advertised driver diameters ranging from 2 to 30 in. For example, the graph shows that a 12-in driver can be no more than 2.5% efficient at 40 Hz, no matter how you manipulate its parameters! The graph emphasizes the point that if you want high efficiency at low frequencies you need a large diameter radiator or equivalently an array of smaller diameter radiators having the combined area of the larger radiator.

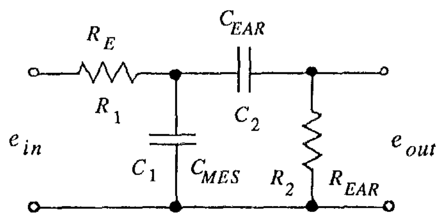

Fig. 12. Analogous circuit of Fig. 2 with infinite suspension compliance. This situation differs from Fig. 7 by the addition of the driver moving mass component, C_{MES}. It corresponds to a driver with only moving mass and voice-coil resistance, radiating into a half-space acoustic load.

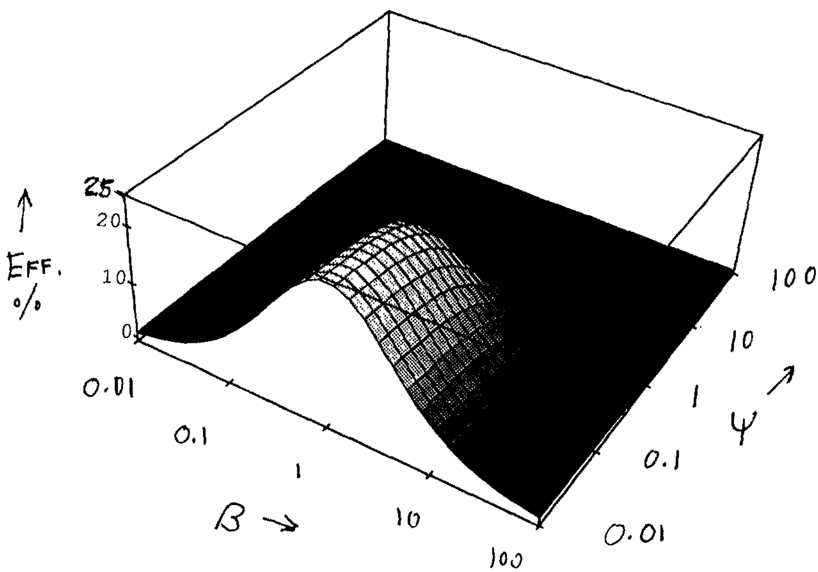

Fig. 13a. Three-dimensional plot of direct-radiator peak efficiency vs β (= R_E/R_{EAR}), and ψ (=C_{MES}/C_{EAR}), using Eq. (46), which is based on the analogous circuit of Fig. 12. A maximum peak efficiency of 25% is reached at $\beta = 1$, and $\psi = 0$. Log scales covering 0.01 to 100 units are used for both β and ψ. Efficiency is plotted in the vertical direction and covers a range of 0 to 25%. This plot emphasizes that high efficiencies are only attained when the moving mass to air-load mass ratio is low ($\psi < 1$).

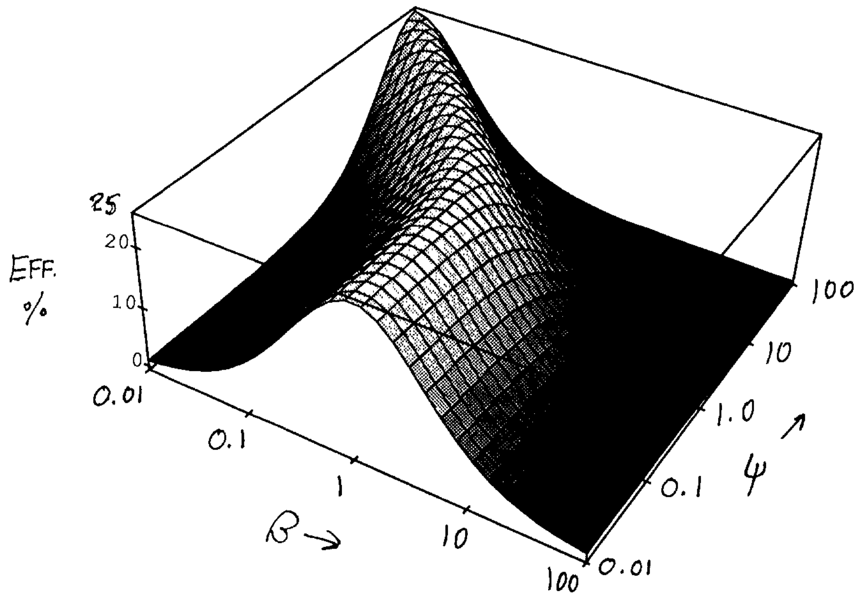

Fig. 13b. Three-dimensional plot of direct-radiator peak efficiency vs β and ψ, using Eq. (46), but multiplied by the factor $1+\psi$, to show the trajectory [$\beta = 1/(1+\psi)$] of the maximum in the $\psi\,\beta$ plane.

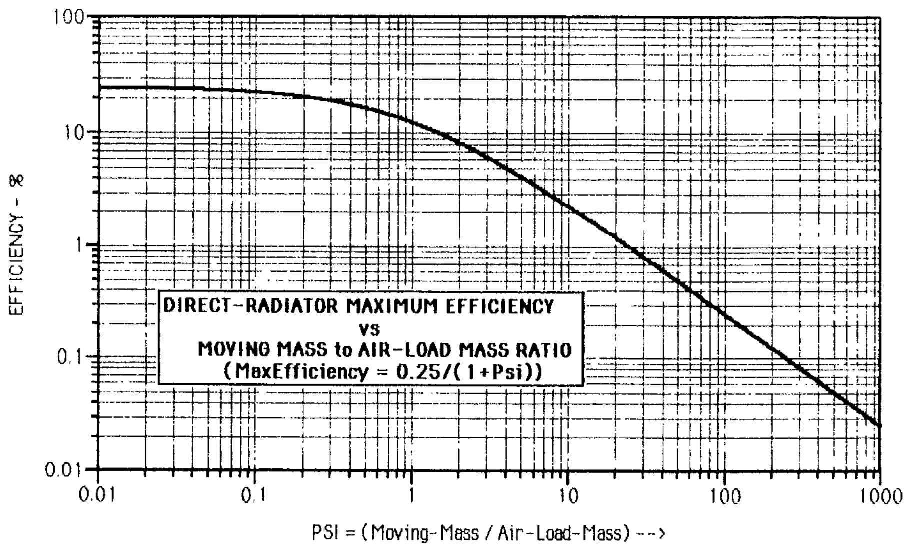

Fig. 14. Plot of direct-radiator maximum efficiency as a function of the moving mass to air-load mass ratio ψ. The plot starts at an efficiency of 25% and then at $\psi = 1$ starts to smoothly decrease 3 dB for each doubling of ψ. Again, this plot emphasizes that high efficiencies are only attained when the moving mass to air-load mass ratio is low ($\psi < 1$)

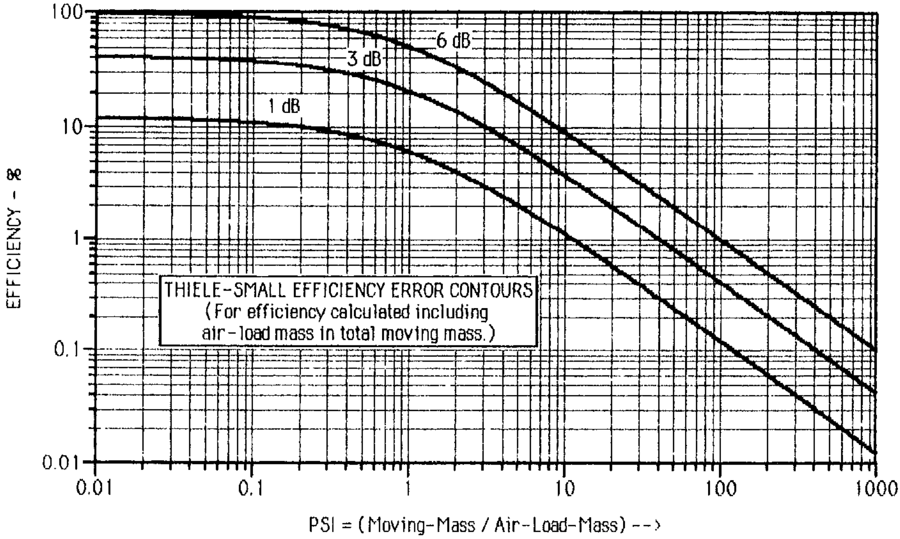

Fig. 15. Error contours, using Eq. (61), comparing the Thiele/Small reference efficiency η_0 with the actual efficiency η_{peak}, as a function of the moving mass to air-load mass ratio ψ. These contours show the regions where the Thiele/Small reference efficiency η_0, which is based on a simplified analogous circuit that neglects the radiation components (but which does include the air-mass load in the total moving mass M_{MS}), is a good approximate of the true efficiency. For example, for an error of 1 dB or less, the Thiele/Small reference efficiency must be below the 1-dB contour line. Note that the Thiele/Small reference efficiency over estimates the actual efficiency, and is *always* larger. Note also, that as ψ increases, η_0 has to be smaller so as not to exceed a specific error contour. For typical driver moving mass to air-load mass ratios in the range of 10 to 20, the Thiele/Small efficiency must be less than about 0.6%, to limit over-estimation errors to 1 dB or less.

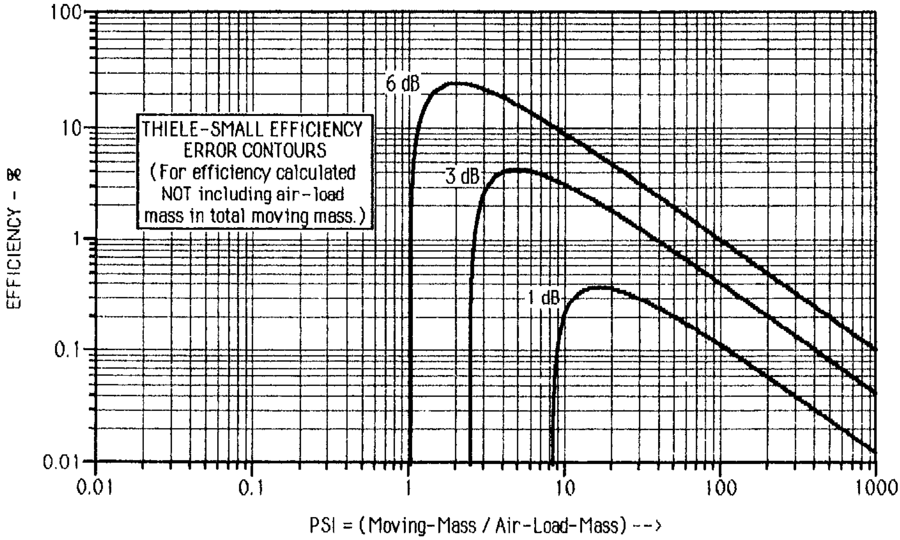

Fig. 16. Error contours as in Fig. 15, but excluding air-load mass in calculation of the Thiele/Small reference efficiency η_0, using Eq. (62). The severely restricted error contour regions in this figure, as compared to Fig. 15, illustrate the necessity of including the air-load mass in the total moving mass, when using the Thiele/Small reference efficiency equation Eq. (54). Note the large-size excluded regions on the left of the graph, for ψ values smaller than the points where the error contours dive to zero, which create much smaller regions where the error is within bounds.

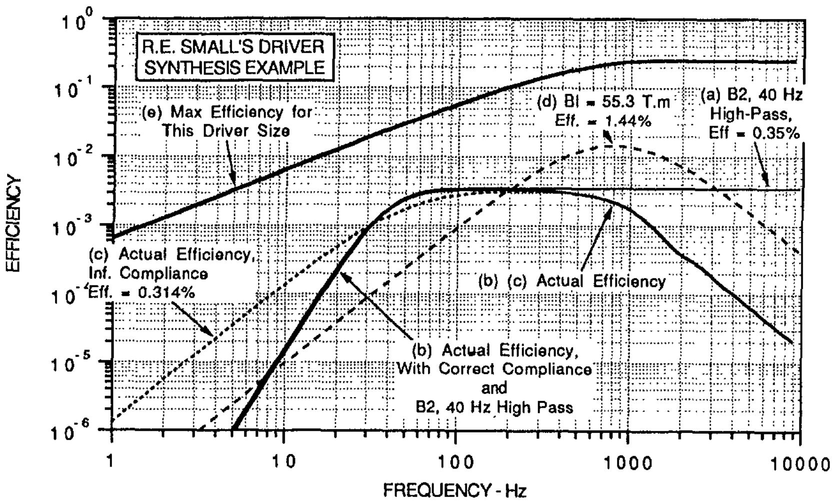

Fig. 17. Various efficiency vs frequency responses corresponding to different conditions in the example Section 7, which is based on one of R. E. Small's example driver designs. This is a 12-inch diameter driver for use in a closed-box air-suspension loudspeaker system, having a second-order Butterworth *(B2)* high-pass response at 40 Hz, in a 2 ft^2 enclosure, with a compliance ratio of 5.

(a) Target response of original design, a *B2* 40-Hz high-pass response. (b) Actual efficiency response (with correct driver and box compliance), predicted in this paper, with the effects of air load included. (c) Actual efficiency response but with infinite driver and box compliance. (d) Efficiency vs frequency response, using Eq. (44), with the *Bl* product set to the value that maximizes efficiency according to Eq. (63) (*Bl* product raised from 14 to 55.3 $T{\cdot}m$). (e) Maximum efficiency curve for a 12-inch diameter driver, calculated using Eq. (28).

Note that target response (a), using the Thiele/Small approximations, and the actual response (b), are very close together below 500 Hz. Above 500 Hz, the actual response rolls off at 6-dB per octave while the target response continues at the 0.35% plateau level. Note also that the maximum efficiency alignment (d), has a very narrow response range that is centered at a frequency much higher than the desired 40-Hz low-frequency cutoff.