Effective Performance of Bessel Arrays*

D. (DON) B. KEELE, JR.**

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The Bessel array is a configuration of five, seven, or nine identical loudspeakers in an equal-spaced line array that provides the same overall polar pattern as a single loudspeaker of the array. The results of a computer simulation are described, which uses point sources to determine the effective operating frequency range, working distance, efficiency, power handling, maximum acoustic output, efficiency–bandwidth product, and power–bandwidth product of the array. The various Bessel configurations are compared to one-, two-, and five-source equal-spaced equal-level equal-polarity line arrays.

As compared to a single source, a five source Bessel array is 14% (0.6 dB) more efficient, can handle 3.5 (+5.4 dB) more power, and has 4 times (+6 dB) the maximum midband acoustic output power, and is usable for omnidirectional radiation up to the frequency where the overall length is 11 wavelengths long. As compared to a two-source equal-level in-phase array, a five-source Bessel array is 43% (2.4 dB) less efficient, can handle 1.75 (+2.4 dB) more power, has the same maximum midband acoustic output power, and is usable for omnidirectional radiation 10 times higher in frequency. A working distance of 20 times the length of the Bessel array was assumed, with the length of the Bessel array (center-to-center distance of outside sources) being four times that of the two-source array. Analysis reveals that the three Bessel arrays have equal maximum acoustic output, but that the five-element Bessel array has the highest efficiency and power–bandwidth product. The seven- and nine-source Bessel arrays are found to be effectively unusable, as compared to the five-source array, due to much lower efficiency, requirement for more sources, and poor high-frequency performance. Judging polar peak-to-peak ripple and high-frequency response, the performance of the Bessel array is found to improve in direct proportion to the working distance away from the array. Unfortunately the phase versus direction and phase versus frequency characteristics of the Bessel array are very nonlinear and make it difficult to use with other sources.

0 INTRODUCTION

The Bessel array is a patented configuration [1] of equally spaced identical transducers, which is said to provide the same overall polar pattern as the polar pattern of a single transducer of the configuration. It is a method that extends the directional operating bandwidth of an array of transducers up into the region where the length of the array is a large number of wavelengths [2], [3]. The introduction to the Philips paper describes the justification for the Bessel configuration [2]:

> An array of N loudspeakers connected in parallel and in phase can radiate N^2 times as much power as a single loudspeaker at very low frequencies, but only N times as much at high frequencies. The power response of the array is therefore quite different from that of the single loudspeakers that compose it. This is due to the increased directivity of the array; whereas the radiation pattern of a single loudspeaker is reasonable omnidirectional, usually up to at least a few kilohertz, that of an array is so only at low frequencies. At high frequencies it becomes much more directive; moreover, the directivity varies considerably with frequency. . . . These shortcomings can be remedied, at some expense to power radiation, by correctly proportioning the drive to the individual speakers of the array. The required proportioning coefficients are based on the Bessel functions.

The configuration normally takes the form of a five-, seven-, or nine-element line array or a 25 (5 × 5)-element symmetrical planar (panel) source. Only the line array Bessel configurations are analyzed in this study. The method used to set the drive levels of the

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** Now also with DBK Associates, Elkhart, IN 46517, USA.

transducers in the array essentially randomizes the polarity of each of the elements [4], [5]. These polarity reversals reduce the sensitivity and the efficiency of the resultant array dramatically as compared to an equal-drive-level equal-polarity array. However, the chosen drive levels do extend the directional operating bandwidth of the array up into the region where the array is many wavelengths long.

To my knowledge, most (if not all) of the available references to the Bessel array contain hardly any information on the effective operation of the configuration. Some questions that immediately come to mind are: How high in frequency does the array operate? How far away from the array must you be? How do the efficiency, power handling, and maximum acoustic output compare to those of other array configurations? Which of the three array types, five-, seven-, or nine-element Bessel, is the best?

These and other questions are answered in this paper by analyzing the Bessel configuration using simulations based on arrays of point sources. The point source, being omnidirectional, should provide omnidirectional radiation when arrayed in a Bessel configuration. The degree to which the analyzed configurations provide omnidirectional coverage is the basis for evaluating their effective performance.

1 REVIEW OF BESSEL-DERIVED SOURCE LEVELS

Quoting again from [2]:

> Consider an array of $2N + 1$ speakers equidistantly spaced in a straight line and driven by a common signal multiplied by coefficients $(a_{-N}, a_{-N+1}, \ldots, a_0, \ldots, a_{N-1}, a_N)$ peculiar to each speaker. Assume that
>
> - The point of observation P is in the far-field region of each speaker.
> - The radiation of each speaker is not influenced by the others.
> - All speakers have the same frequency and directional response $A(\omega, \theta)$.

The required proportioning of the drive levels (both level and polarity) of each of the transducers of the configuration is based on numbers derived from the Bessel function of the first kind and order n[2], [6]:

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\,(n+k)!} \; . \tag{1}$$

The method relies on a mathematical property of the Bessel function, which is

$$\left|\sum_{n=-\infty}^{\infty} J_n(z)\right| = \left|\sum_{n=-\infty}^{\infty} J_n(z)\, e^{(jnx)}\right| = \left|e^{(jz \sin x)}\right| = 1 \; . \tag{2}$$

This property, combined with the equation that gives the pressure magnitude and phase at point P, at a particular frequency and angle for an array of sources (at an infinite distance),

$$p(\omega, \theta) = A(\omega, \theta) \sum_{n=-N}^{N} a_n\, e^{(-jnx)} \tag{3}$$

where

x = $\omega l \sin \theta / c$ (assumes sample point at infinite distance)
ω = frequency, rad/s, = $2\pi f$
c = velocity of sound
l = distance between loudspeakers
θ = angle between sample point vector and array axis
$A(\omega, \theta)$ = amplitude–phase function giving directional characteristics of a single source
a_n = drive level of source n giving strength and polarity

yields a function that makes the dependence of the magnitude of p on direction and frequency the same for the array as for a single loudspeaker that makes up the array:

$$\begin{aligned} p(\omega, \theta) &= A(\omega, \theta) \sum_{n=-N}^{N} J_n(z)\, e^{(-jnx)} \\ &= A(\omega, \theta)\, e^{(-jz \sin x)} \, , \qquad N \to \infty \end{aligned} \tag{4}$$

or

$$|p(\omega, \theta)| = |A(\omega, \theta)| \; .$$

Eq. (4) clearly shows that the polar pattern of the array will be the same as that of one of the sources that make up the array. This function only works exactly, of course, for an infinite array of sources and a sample point at an infinite distance from the array. A finite-sized array of five, seven, or nine sources is also found to work quite well even if the drive levels are restricted to approximate values limited to the integer ratios ±1 and ±2 (±0.5 and ±1 in practice). These approximate values allow the drive levels of the array to be set by simple series–parallel connections of the drivers.

The approximate coefficient values are derived from the Bessel function by searching for arguments (both integer and noninteger values of z are allowed) that yield a coefficient ratio series that can be approximated by ±1 and ±2. An argument value of $z = 1.5$ is found to be a good choice for the five-element array coefficients. Fig. 1 shows the resultant coefficient values of $J_n(1.5)$ over the range $-10 \leq n \leq +10$, plotted in bar graph form. Both the actual values and the absolute values are plotted for comparison purposes. The plotted values show that the function decreases very rapidly to very small values for n beyond ±3. Truncating the series beyond these values eliminates relatively little from the sum.

Choosing the range $-2 \leq n \leq 2$ yields the series

$$J_{-2}(1.5),\ J_{-1}(1.5),\ J_0(1.5),\ J_2(1.5),\ J_3(1.5)$$

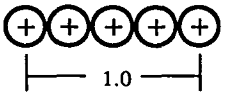
$$= 0.232,\ -0.558,\ 0.512,\ 0.558,\ 0.232$$

the ratios of which, can be approximated as follows:

five-element Bessel ratios

$$= +0.5 : -1 : +1 : +1 : +0.5$$

configuration:

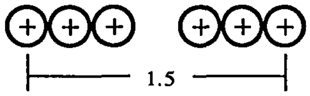


Likewise, the corresponding series for $z = 2.405$ and $z = 3.83$ yield the approximate drive ratios for the seven- and nine-element Bessel arrays:

seven-element Bessel ratios

$$= -0.5 : +1 : -1 : 0 : +1 : +1 : +0.5$$

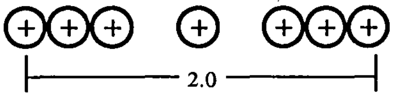
configuration:



nine-element Bessel ratios

$$= +0.5 : -1 : +1 : 0 : -1 : 0 : +1 : +1 : +0.5$$

configuration:



Note that sources that have zero drive levels can be eliminated from their respective arrays, making the seven-element Bessel array have six actual sources and the nine-element Bessel array have seven actual sources. Note also that the spaces for the removed sources must be preserved.

These drive ratios can be implemented by simple series–parallel hookups for each of the three configurations. Each ratio combination can be connected in a mostly parallel or a mostly series hookup. Only the more practical mostly parallel connection is analyzed here.

2 SIMULATION METHODS

The polar and frequency response simulations were accomplished by evaluating a more complete version of Eq. (3), which takes proper account of the actual distance from each source to the sample point, with no approximations. This more complete equation allows a proper evaluation of the effective working distances from the array. The equation appears as

$$p(\omega, \theta) = A(\omega, \theta) \sum_{n=-N}^{N} \frac{a_n}{r_n} e^{(-jkr_n)} \qquad (5)$$

where

k = wavenumber, $= \omega/c = 2\pi f/c = 2\pi/\lambda$
ω = frequency, rad/s, $= 2\pi f$
c = velocity of sound
λ = wavelength, $= c/f$
f = frequency, Hz
r_n = distance from source n to sample point, $= |r_n|$
a_n = strength and polarity of source n.

All distances in this paper are referenced to a system that has a unit velocity of transmission. This means that at a frequency of 1 Hz a unit distance is 1 wavelength

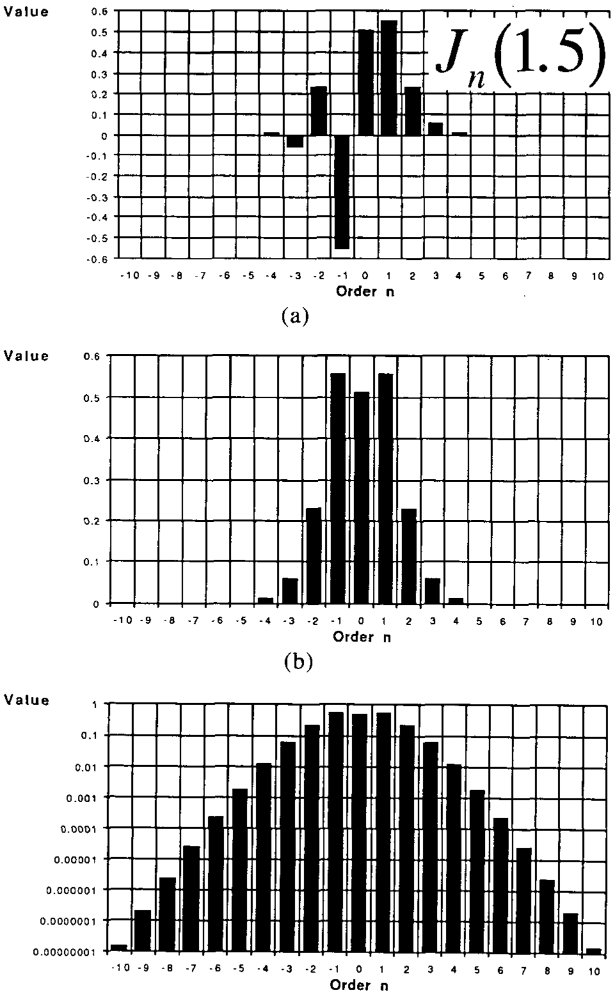


Fig. 1. Bar graphs of Bessel function of first kind and order n for argument of 1.5 [$J_n(1.5)$]; $-10 \leq n \leq +10$. (a) Linear vertical scale. (b) Absolute values using linear vertical scale. (c) Absolute values using logarithmic vertical scale, ranging from 10^{-8} to 1.0. Note that values grow rapidly very small for n beyond ± 3. Values in the range of $|n| \leq 2$ are used to generate source drive levels and polarities for five-source Bessel array ($+0.5 : -1 : +1 : +1 : +1 : +0.5$).

long. All the five-element arrays have a unit overall source center-to-center length. A working distance of 20 units implies that the pressure sampling point is 20 times the length of the five-element array away from the center of the array.

The array was oriented so that its axis was along the y axis, with its center at the origin of the coordinate system. Rotation was always around the center of the array (the origin), with the zero angle in the direction of the positive x axis and positive angles in the counterclockwise direction. For the Bessel arrays, the sources with higher number Bessel coefficients were above the x axis (positive angles).

A number of the analysis factors require the calculation of the peak-to-peak ripple, in decibels, for the polar directional pattern at a specific frequency. The equation to calculate this ripple factor [2] is

$$\text{peak-to-peak ripple [dB]} = R(\omega) = 20 \log \left(\frac{|p(\omega, \theta)|_{\max}}{|p(\omega, \theta)|_{\min}} \right)_{-\pi \leq \theta \leq \pi} . \tag{6}$$

All calculations for this paper were done on a Macintosh II desktop computer using a combination of the Microsoft spreadsheet program Excel and the math analysis program Mathematica by Wolfram Research, Inc. (used for all graphic output).

3 ARRAY ANALYSIS FACTORS

The various Bessel configurations (five-, seven-, and nine-source line arrays) were compared to one-, two-, and five-source equal-spaced equal-level equal-polarity line arrays. A number of analysis factors were used in the comparison: voltage sensitivity, impedance, efficiency, maximum input power handling, maximum acoustic output power and SPL, maximum operating frequency, working distance, polar directional response, frequency response (both magnitude and phase), polar peak-to-peak ripple versus frequency, efficiency–bandwidth product, power–bandwidth product, and power–bandwidth product per unit. These analysis factors are now described individually.

3.1 Voltage Sensitivity

The voltage sensitivity of a system is the on-axis sound pressure level (SPL) generated at a specific distance for a particular applied voltage. In this paper all measurements are referenced or normalized to a point source that is assumed to have all unit specifications, that is, a sensitivity of 1- or 0-dB SPL for an applied unity voltage, at a 1-unit distance. The sensitivity of the analyzed arrays is simply the total of the individual drive levels.

3.2 Impedance

The input electrical impedance for each analyzed array was computed assuming a unity impedance (resistance) for each of the individual sources.

3.3 Efficiency

The electroacoustic efficiency (electric input power divided by the resultant acoustic output power) of each array was computed by direct comparison to a single point source, in the omnidirectional radiation region of the array's frequency range. An efficiency of unity was assigned to the point source. The efficiency of an array was computed by squaring its sensitivity and multiplying by its impedance:

$$\eta_0 = \frac{P_{\text{out}}}{P_{\text{in}}} = \text{Sens}^2 Z_{\text{in}} . \tag{7}$$

3.4 Power Handling

The maximum input electric power handling of an array was computed by summing the individual source powers computed by applying a unity input voltage and assuming unity impedances for all individual sources.

3.5 Maximum Acoustic Output Power and SPL

The maximum acoustic output power was computed by multiplying the array's efficiency by its maximum input electric power,

$$P_{\text{out}} = \eta_0 P_{\text{in}} \tag{8}$$

The sound pressure level, in decibels, was calculated by using $10 \log_{10}(P_{\text{out}}/P_{\text{ref}})$, where P_{ref} is unity (the power output of a single source).

3.6 Maximum Operating Frequency

The maximum operating frequency was assessed by simulating the polar response of the analyzed array using Eq. (5) and then finding the maximum frequency up to which the peak-to-peak polar magnitude ripple [Eq. (6)] did not exceed a specific amount, usually 3, 4, 6, or 9 dB. Note that a point source has a maximum operating frequency of infinity, using this definition.

3.7 Working Distance

The working distance was assessed similarly to the maximum operating frequency by polar simulations and then noting the minimum operating distance that provided a specific peak-to-peak polar ripple. Usually a specific working distance (in terms of array lengths of 5, 10, or 20 units) was chosen, and then all the relevant parameters were calculated.

3.8 Polar Response

Polar directional responses were computed using Eq. (5) at various angles and distances for each of the analyzed configurations. A set of polar responses at a fixed working distance (usually 20 units) at various frequencies were simulated along with a set at a fixed frequency (usually 10 Hz) at various working distances. Both magnitude and sometimes phase versus angle plots are displayed. The linear phase effects of transport delay between source and sample point were removed in all phase displays.

3.9 Frequency Response (Magnitude and Phase)

Magnitude and phase frequency responses were computed using Eq. (5) at various angles and distances for each of the analyzed configurations. A set of frequency responses at various angles at a fixed distance (usually 20 units) were simulated along with a set at a fixed angle (usually 45°) at various working distances. Both magnitude and sometimes phase (also group delay in one case) versus frequency plots are displayed. The linear phase effects of transport delay between source and sample point were removed in all phase displays.

3.10 Polar Peak-to-Peak Ripple versus Frequency

A plot of the polar peak-to-peak ripple, in decibels, versus frequency indicates the tradeoff of polar nonlinearities versus the high-frequency limit. In general, all arrays exhibit increasing polar ripple as the frequency is increased.

3.11 Efficiency–Bandwidth Product

The efficiency–bandwidth product was computed by forming the product of the efficiency and the maximum operating frequency. This number gives a comparative value that indicates how thrifty the analyzed array is in terms of its efficiency and operating frequency range.

3.12 Power–Bandwidth Product

The power–bandwidth product was computed by forming the product of the maximum acoustic output power and the maximum operating frequency. This number gives a comparative value, which indicates how well the analyzed array functions in terms of its output power and operating frequency range.

3.13 Power–Bandwidth Product per Unit

The power–bandwidth product per unit was computed by dividing the power–bandwidth product by the number of units in the array. This number can be thought of as a figure of merit for comparing the operating effectiveness of the analyzed arrays on a per-unit basis.

4 SIMULATION RESULTS

Several different point source configurations were analyzed and compared for this study. All configurations were analyzed in terms of the performance of a single point source. The following configurations were analyzed.

1) Two equal-level equal-polarity sources with center-to-center spacing of 0.25 unit (same center-to-center spacing as the individual spacing of the five-element arrays)

2) Two equal-level equal-polarity sources with center-to-center spacing of 1.0 unit (same overall center-to-center length as the five-element arrays)

3) Five equal-level equal-polarity equal-spaced sources with individual center-to-center spacing of 0.25 unit and overall center-to-center length of 1.0 unit

4) Five-source Bessel array with individual center-to-center spacing of 0.25 unit and overall center-to-center length of 1.0 unit

5) Seven-source Bessel array with individual center-to-center spacing of 0.25 unit and overall center-to-center length of 1.5 units

6) Nine-source Bessel array with individual center-to-center spacing of 0.25 unit and overall center-to-center length of 2.0 units.

Note that all the arrays have the same individual center-to-center spacings (except for the two-source configuration with center-to-center length of 1.0 unit). This means that the overall array length increases in direct proportion to the number of sources. This models the real-world situation of using the same-size transducers packed as close together as possible.

For each configuration, several possible analysis factors were calculated: voltage sensitivity, impedance, efficiency, maximum input power handling, maximum acoustic output power and SPL, maximum operating frequency, working distance, polar directional response, frequency response (magnitude, phase, and group delay), efficiency–bandwidth product, power–bandwidth product, and power–bandwidth product per unit. Further explanations of these factors are given in Sec. 2. The results of the simulations are described in the following sections and shown in Figs. 2 to 32.

4.1 Single Point Source

A single point source is the reference for all the following array configurations. The single point source is arbitrarily assigned all unit parameters and its characteristics are shown in Table 1. Note that all the frequency-dependent factors have infinite values because the point source by definition has no upper frequency limit.

The polar response of the point source (not shown) is a perfect circle, while its frequency and phase responses (not shown) are straight lines. The polar and frequency responses of the reference point source are not distance dependent. Note that table entries have been reserved for working distances of 5, 10, and 20 units at peak-to-peak ripple values of 3, 4, and 6 dB.

4.2 Two Sources, Equal Level, Equal Polarity, with 0.25- and 1.0-Unit Center-to-Center Spacings

The two-source array is the simplest configuration, one step above the single source, and is used quite frequently to increase the acoustic output as compared to a single source. Unfortunately, as the following simulations show, the maximum frequency of operation drops dramatically because of source interference and lobing. Two double-source configurations, with center-to-center spacing of 0.25 and 1.0 unit, were analyzed and are described in the following section.

The two-source array with 0.25-unit center-to-center spacing has the same center-to-center spacing as the individual spacing of the five-element arrays. This close side-by-side spacing is the logical configuration for getting the most performance (highest operating bandwidth) from a two-source array. All the characteristics and calculated parameters for the two-source array with 0.25-unit center-to-center spacing are shown in Table 2.

The 1.0-unit center-to-center spaced two-source array has the same center-to-center spacing as the overall center-to-center spacing (outside sources) of the five-element arrays. If you just simply remove the center three sources of the five-element array, you get this spacing. All the characteristics and calculated parameters for the two-source array with 1.0-unit center-to-center spacing are shown in Table 3.

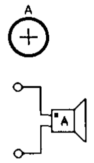
All the responses and characteristics for the 1.0-unit center-to-center spacing array are the same as those for the 0.25-unit center-to-center spacing array, but shifted down in frequency by two octaves (frequency × ¼). The data on the 1.0-unit center-to-center spacing array have been included for comparing against the five-source arrays, which have the same length.

4.2.1 Polar Responses

The polar magnitude responses of the two-source array with 0.25-unit center-to-center spacing, at constant distance, are shown in Fig. 2. The polars are displayed at half-decade intervals from 0.316 to 31.6 Hz and at a working distance of 20 units. An additional polar at 2.0 Hz is also displayed. All the polar plots

Table 1. Array type: single source.

Configuration:



Number Units (N):	1	
Overall Length (c-c):	0	
Strengths:	1	
Impedance (Z_{in}):	1	(0.0 dB)
Voltage Sensitivity:	1	(0.0 dB)
Efficiency(η_o):	1	(0.0 dB)
Maximum Input Power (P_{in}):	1	(0.0 dB)
Maximum Acoustic Output Power (P_{out}):	1	(0.0 dB)
Maximum Sound Pressure Level:	1	(0.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Power-Bandwidth Product (P_{out} x F_{max}):

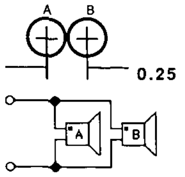
Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Table 2. Array type: two sources (L = 0.25 unit), equal level, same polarity.

Configuration:



Number Units (N):	2	
Overall Length (c-c):	0.25	
Strengths:	1 : 1	
Impedance (Z_{in}):	0.5	(-3.0 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	2	(+3.0 dB)
Maximum Input Power (P_{in}):	2	(+3.0 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	1.00
4	1.10
6	1.30

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	2.00
4	2.20
6	2.60

Power-Bandwidth Product (P_{out} x F_{max}):

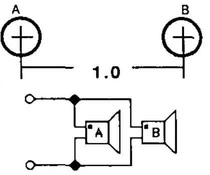
Distance ->	5, 10, 20
Ripple (dB): 3	4.00
4	4.40
6	5.20

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5, 10, 20
Ripple (dB): 3	2.00
4	2.20
6	2.60

Table 3. Array type: two sources (L = 1.0 unit), equal level, same polarity.

Configuration:



Number Units (N):	2	
Overall Length (c-c):	1.0	
Strengths:	1 : 1	
Impedance (Z_{in}):	0.5	(-3.0 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	2	(+3.0 dB)
Maximum Input Power (P_{in}):	2	(+3.0 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	0.25
4	0.28
6	0.33

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	0.50
4	0.55
6	0.65

Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	1.00
4	1.10
6	1.30

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5, 10, 20
Ripple (dB): 3	0.50
4	0.55
6	0.65

displayed in this paper cover a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that the on-axis level is 0 dB. Note that at 2 Hz, where the sources are one-half wavelength apart, the first polar null at 90° off axis occurs. Note also that above about 1.8 Hz, the polar response is multilobed and hence unusable for omnidirectional response.

The polar responses for the 0.25-unit center-to-center spacing array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 3. Polars at distances of 1.25, 2.5, 5, 10, and 100 000 units are shown. Observe that the polar responses essentially exhibit no change with increasing working distances beyond about 2.5 units (10 times array length). Note that the 100 000-unit distance is extremely far from the array; essentially an effective infinity. If the overall length of the array were 2 ft (0.6 m), this distance would be about 38 mi (60 km) away.

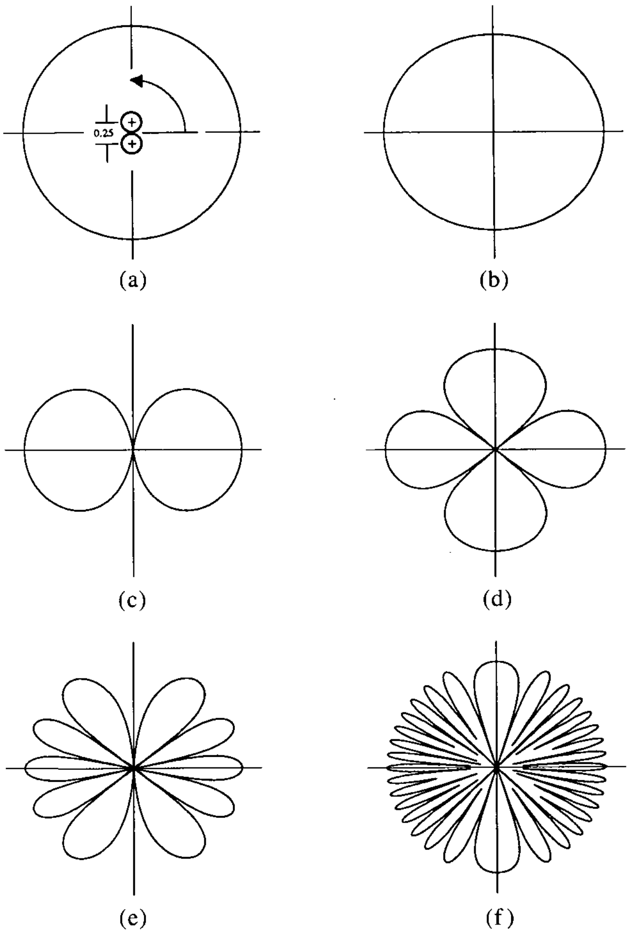


Fig. 2. Polar magnitude responses for two-source equal-level equal-polarity array with 0.25-unit center-to-center spacing at a constant working distance of 20 units. Spacing is the same as individual center-to-center spacing of five-source arrays. Polars are displayed at half-decade intervals from 0.316 to 31.6 Hz, with additional polar at 2.0 Hz. (a) 0.316 Hz. (b) 1 Hz. (c) 2 Hz. (d) 3.16 Hz. (e) 10 Hz. (f) 31.6 Hz. Polar plot covers a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that on-axis level is 0 dB. Sources are one-half wavelength apart at 2 Hz and exhibit a null at ±90° off axis (c). Note that polar response is mostly omnidirectional at and below 1 Hz, but gets progressively narrower and gains additional lobes as frequency increases.

Only a few polar responses were done on the two-source array with 1.0-unit center-to-center spacing, mainly to illustrate the variation of phase versus angle with frequency and working distance. The previous two-source array exhibits the same behavior but is four times higher in frequency. Note that the first polar null at 90° off axis occurs at a frequency of 0.5 Hz, where the sources are one-half wavelength apart (not shown).

Fig. 4 shows various magnitude and phase polar responses at different frequencies f and distances D for the two-source array with 1.0-unit center-to-center spacing. The following four combination are plotted:

1) $f = 1$ Hz, $D = 20$ units
2) $f = 1$ Hz, $D = 100\,000$ units
3) $f = 2$ Hz, $D = 20$ units
4) $f = 2$ Hz, $D = 100\,000$ units.

The phase versus direction plots show the phase of the pressure at the sample point versus the off-axis direction. The phase values are referenced to the input signal of the array. The effects of linear phase lag and delay due to sample distance have been removed in

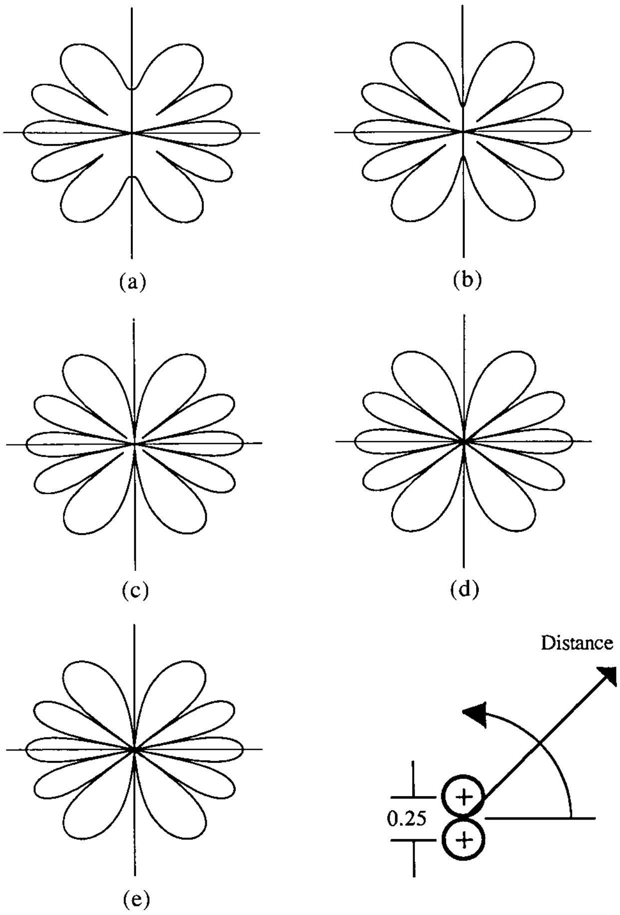


Fig. 3. Polar magnitude responses for two-source equal-level equal-polarity array with 0.25-unit center-to-center spacing at fixed frequency of 10 Hz and different working distances. (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 100 000 units. Observe that polar responses essentially exhibit no change with increasing working distance beyond about 2.5 units (10 times array length).

this plot and in all the polar and frequency response plots of this paper. Note that the phase values switch between 0 and ± 180°, depending on the polar lobe on which the pressure sample point happens to be. The phase always starts out at 0° (on axis). At distances far from the array, the phase transitions occur very abruptly according to the angle, with no rounded corners.

4.2.2 Frequency Responses

The magnitude versus frequency responses of the two-source array with 1.0-unit center-to-center spacing at constant distance are shown in Fig. 5. The responses are shown at angles ranging from 0 to +90°, with steps of 15°, at a working distance of 20 units. Note that the response gets progressively rougher as the angle increases due to the nulls in the response moving down in frequency. The magnitude versus frequency responses at a fixed angle of 45° and at different working distances are given in Fig. 6. Distances of 1.25, 2.5, 5, 10, and 100 000 units are shown. Note that the frequency response changes very little with distance beyond 5 units.

The frequency range of the responses goes from 0.1 to 10 Hz with a log frequency scale. Note that the frequency scale is marked with decade number (log *f*) rather than frequency (−1.0 = 0.1 Hz, 0.0 = 1 Hz, and so on).

To illustrate the variation of phase versus frequency and working distance, several magnitude and phase responses were done on the two-source array with 1.0-unit center-to-center spacing. Fig. 7 shows these responses with a fixed angle of 45° and distances of 5 and 100 000 units. Observe that the phase again is either 0 or ± 180°, depending on the polar lobe in which the sample point happens to be. This phase versus frequency behavior looks suspiciously nonminimum phase, but is actually minimum phase [7]. This comment only applies to the two-source array, however, where the response at the sample point is strictly due to a signal plus a single delayed signal of reduced amplitude.

4.2.3 Polar Peak-to-Peak Ripple versus Frequency

Fig. 8 shows a plot of the polar peak-to-peak ripple, in decibels, versus frequency for both two-source arrays

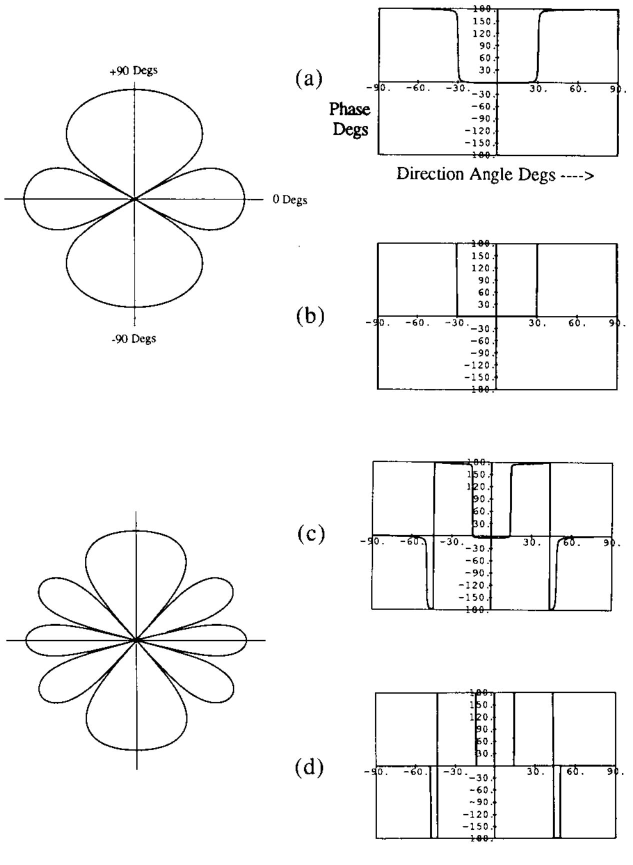


Fig. 4. Polar magnitude and phase responses at various frequencies and distances for two-source equal-level equal-polarity array with 1.0 unit center-to-center spacing. Phase versus direction plots show phase of pressure at sample point versus off-axis direction. (a) f = 1 Hz, D = 20 units. (b) f = 1 Hz, D = 100 000 units. (c) f = 2 Hz, D = 20 units. (d) f = 2 Hz, D = 100 000 units. Phase values are referenced to array input signal. Effects of linear phase lag and delay due to sample distance have been removed. Note how phase changes rapidly from 0 to ± 180° as direction angle increases, as each separate lobe is transversed. Note also that this array is spaced one-half wavelength apart at 0.5 Hz.

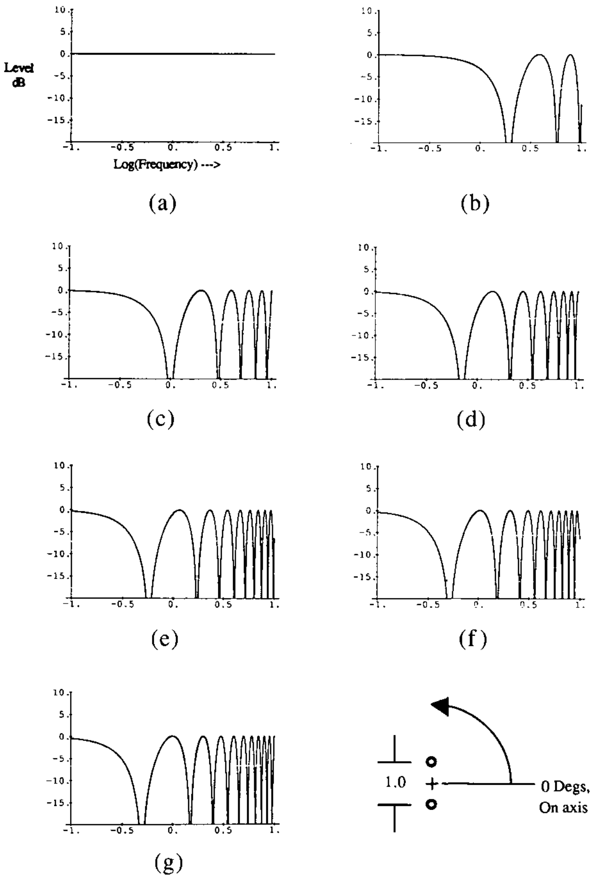


Fig. 5. Magnitude frequency responses of two-source equal-level equal-polarity array with 1.0-unit center-to-center spacing at constant distance of 20 units and frequency range of 0.1 to 10 Hz. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). Responses are shown at angles ranging from 0 to +90°, with steps of 15°. (a) 0°. (b) 15°. (c) 30°. (d) 45°. (e) 60°. (f) 75°. (g) 90°. Note that response gets progressively rougher as angle increases.

at a working distance of 20 units. Note that the ripple increases very rapidly above 0.25 Hz for the 1.0-unit spacing and above 1 Hz for the 0.25-unit spacing.

Fig. 9 shows the polar peak-to-peak ripple versus frequency at several different working distances from 2.5 to 100 000 units for the 1.0-unit spaced two-source array. The graph exhibits essentially no change at distances beyond 5 units.

Fig. 10 shows a polar response of the 0.25-unit spaced array at a distance of 20 units and a frequency of 1.1 Hz, which corresponds to the frequency where the peak-to-peak ripple is 4 dB. Note that the polar response is very smooth, but squashed vertically, and exhibits its maximum deviation (−4 dB) at ±90°. As will be seen, this is a characteristic of all the equal-level equal-phase arrays.

The following approximate equations relate the maximum operating frequency for omnidirectional radiation $f_{\max}$ to the array length for the two-source equal-level equal-polarity arrays.

For the 0.25-unit spaced array,

$$\begin{aligned} f_{\max} &\approx 1.0\,\frac{c}{L}\,, \quad \text{for 3-dB peak-to-peak polar ripple} \\ &\approx 1.1\,\frac{c}{L}\,, \quad \text{for 4-dB peak-to-peak polar ripple} \\ &\approx 1.3\,\frac{c}{L}\,, \quad \text{for 6-dB peak-to-peak polar ripple} \\ &\approx 1.6\,\frac{c}{L}\,, \quad \text{for 9-dB peak-to-peak polar ripple} \end{aligned} \tag{9}$$

For the 1.0-unit spaced array,

$$\begin{aligned} f_{\max} &\approx 0.25\,\frac{c}{L}\,, \quad \text{for 3-dB peak-to-peak polar ripple} \\ &\approx 0.28\,\frac{c}{L}\,, \quad \text{for 4-dB peak-to-peak polar ripple} \\ &\approx 0.33\,\frac{c}{L}\,, \quad \text{for 6-dB peak-to-peak polar ripple} \\ &\approx 0.40\,\frac{c}{L}\,, \quad \text{for 9-dB peak-to-peak polar ripple} \end{aligned} \tag{10}$$

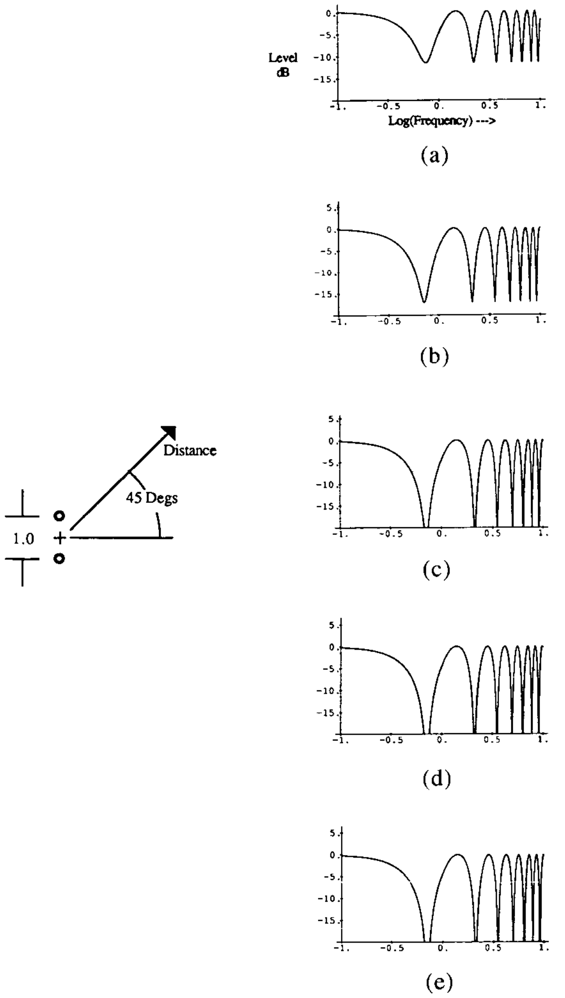


Fig. 6. Magnitude frequency responses at fixed angle of 45° and different working distances for two-source equal-level equal-polarity array with 1.0-unit center-to-center spacing and frequency range of 0.1 to 10 Hz. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 100 000 units. Observe that frequency response changes very little with distance beyond 5 units.

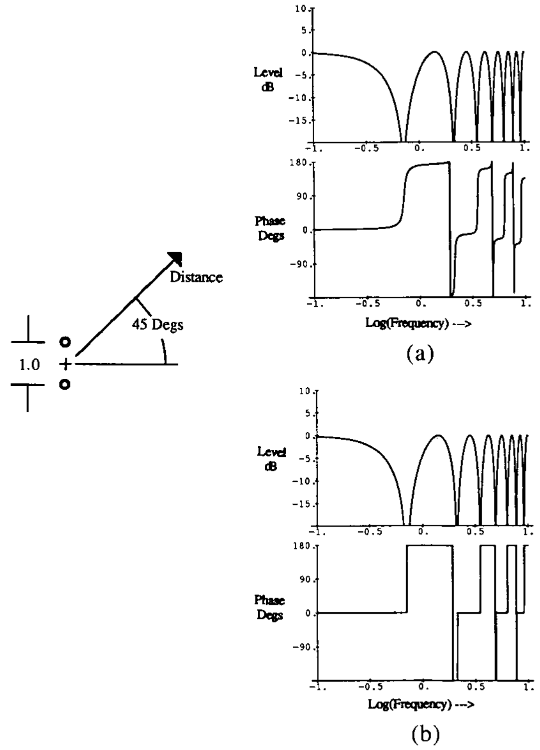


Fig. 7. Off-axis +45° magnitude and phase versus frequency responses for two-source equal-level equal-polarity array with 1.0-unit center-to-center spacing. (a) 5 units. (b) 100 000 units. Observe that phase is either 0 or ± 180°, depending on the polar lobe in which the sample point happens to be. At the farther distance the phase switches very rapidly. Phase versus frequency behavior is near nonminimum phase but is actually minimum phase.

where c is the velocity of sound and L the length of the array (center-to-center distance of sources).

4.2.4 Discussion

At low frequencies, the two-source arrays exhibit mostly omnidirectional behavior below 0.25 Hz for the 1.0-unit spaced array and below 1.0 Hz for the 0.25-unit spaced array. The upper frequency limit for omnidirectional radiation occurs at the frequency where the sources are about one-quarter wavelength apart. Below this frequency, the efficiency is twice that of the single source, while the maximum output is four times that of the single source.

The directional characteristics essentially do not change with working distance beyond a point that is roughly 10 times the length of the array. The behavior of the 1.0-unit spaced two-source array exhibits the same activity as the 0.25-unit spaced two-source array, but at one-fourth the frequency.

The off-axis polar phase alternates between 0 and 180° depending on the polar lobe in which the sample point is. The off-axis phase versus frequency data exhibit the same switching behavior with increasing frequency, but are found to be minimum phase.

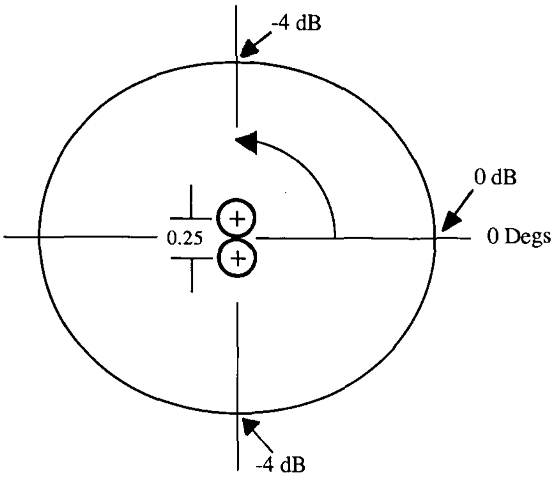


Fig. 10. Magnitude polar response of 0.25-unit center-to-center spaced two-source equal-level equal-polarity array at a distance of 20 units and a frequency of 1.1 Hz. This polar is at the frequency where the peak-to-peak ripple is 4 dB. Note that polar is very smooth, but squashed vertically, and exhibits its maximum deviation (−4 dB) at ±90°. Polar plot covers a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. Polar is normalized so that on-axis level is 0 dB.

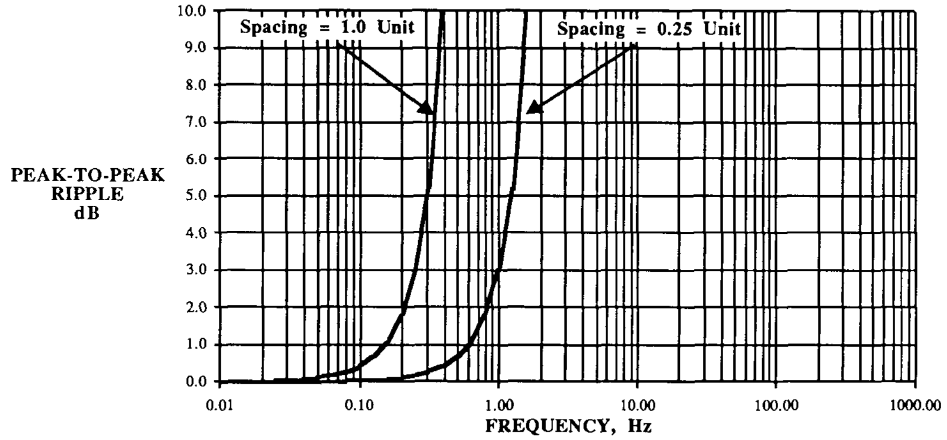


Fig. 8. Polar peak-to-peak ripple versus frequency for both two-source equal-level equal-polarity arrays at a working distance of 20 units. Note that ripple increases very rapidly above 1 Hz for 0.25-unit spaced array and above 0.25 Hz for 1.0-unit spaced array. Velocity of propagation 1 unit/s.

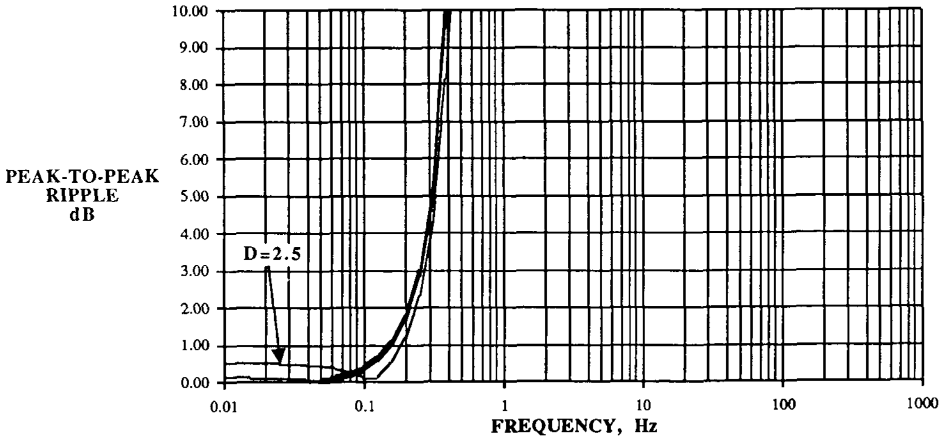


Fig. 9. Polar peak-to-peak ripple, versus frequency at working distances of 2.5, 5, 10, 20, 40, 80, 160, 1000, 10 000, and 100 000 units for 1.0-unit spaced two-source equal-level equal-polarity array. Graph exhibits essentially no change at distances beyond 5 units. Note close bunching of all curves.

4.3 Five Sources, Equal Level Equal Polarity Equal Spaced, with Overall Center-to-Center Length of 1.0 Unit

This array contains five sources equally spaced with equal levels and equal polarities. The overall length, measured from the centers of the outside sources, is 1 unit. The individual source center-to-center spacing is 0.25 unit. The characteristics and calculated parameters for this array are shown in Table 4.

This array provides 25 times the acoustic output power at low frequencies as compared to a single source. This array was included for direct comparison to the five-source Bessel array. The only difference between this array and the Bessel array is the amplitude and polarity of the source drive levels.

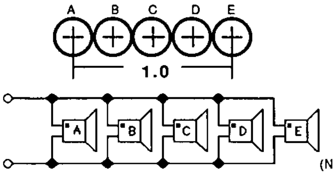
4.3.1 Polar Responses

The polar magnitude responses of the five-source equal-level array, at constant distance, are shown in Fig. 11. They are displayed at half-decade intervals from 0.1 to 10 Hz at a working distance of 20 units. The polar plot covers a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polar plots are normalized so that the on-axis level is 0 dB. The polars of this array are much more complex and directive than the two-source polars (Fig. 2). For omnidirectional radiation, the five-source equal-level array ios unusable above about 0.6 Hz.

The polar magnitude responses for the five-source equal-level array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 12. Polars at distances of 1.25, 2.5, 5, 10, 20, and 100 000 units are shown. Note that the polar response changes very little with distance beyond roughly 10 units (10 array lengths).

Table 4. Array type: five sources, equal level, equal spacing, same polarity.

Configuration:



(Note Polarity Dots!)

Number Units (N):	5	
Overall Length (c-c):	1	
Strengths:	1 : 1 : 1 : 1 : 1	
Impedance (Z_{in}):	1/5 = 0.2	(-7.0 dB)
Voltage Sensitivity:	5	(+14.0 dB)
Efficiency(η_o):	5	(+7.0 dB)
Maximum Input Power (P_{in}):	5	(+7.0 dB)
Maximum Acoustic Output Power (P_{out}):	25	(+14.0 dB)
Maximum Sound Pressure Level:	5	(+14.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	0.35
4	0.40
6	0.48

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	1.75
4	2.00
6	2.40

Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->	5, 10, 20
Ripple (dB): 3	8.75
4	10.00
6	12.00

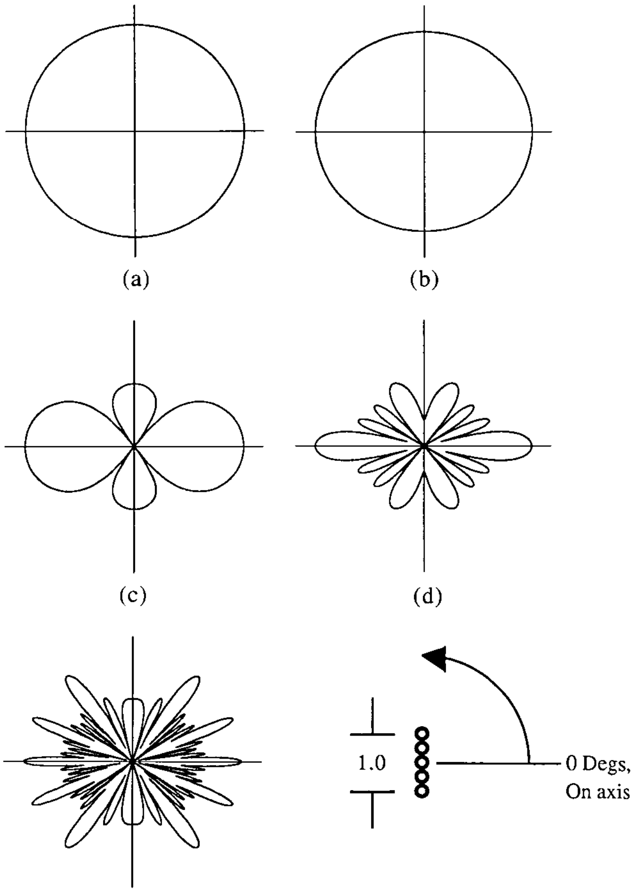
Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5, 10, 20
Ripple (dB): 3	1.75
4	2.00
6	2.40

Fig. 13 shows various magnitude and phase polar responses at different frequencies *f* and distances *D* for the five-source array. The following five combinations are plotted:

1) f = 0.5 Hz, D = 20 units
2) f = 1 Hz, D = 20 units
3) f = 1 Hz, D = 100 000 units
4) f = 2 Hz, D = 20 units
5) f = 2 Hz, D = 100 000 units.

The phase versus direction plots show the phase of the pressure at the sample point versus the off-axis direction. The effects of the linear phase delay due to sample distance have been eliminated. Note that the phase values switch between 0 and +180° depending on the polar lobe on which the pressure sample point happens to be. The phase always starts out at 0° (on


(a) (b)

(c) (d)

(e)

Fig. 11. Polar magnitude responses of five-source unit-length equal-level equal-polarity equal-spaced array at a constant working distance of 20 units. Polars are displayed at half-decade intervals from 0.1 Hz to 10 Hz. (a) 0.1 Hz. (b) 0.316 Hz. (c) 1 Hz. (d) 3.16 Hz. (e) 10 Hz. Note how directive and complex polars get above 0.316 Hz. Polar plot covers a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that on-axis level is 0 dB.

axis). At distances far from the array, the phase changes occur more abruptly with angle. The phase variation with direction for the five-source array is very similar to the behavior of the two-source arrays.

4.3.2 Frequency Responses

The magnitude versus frequency responses of the five-source equal-level array at constant distance are shown in Fig. 14. The responses are plotted at angles ranging from 0 to 90°, with steps of 15° at a working distance of 20 units. Note that the response gets progressively rougher as the angle increases, similarly to the two-source arrays.

The magnitude versus frequency responses at a fixed angle of 45° and at different working distances are plotted in Fig. 15. Distances of 1.25, 2.5, 5, 10, and 100 000 units are shown. Note that the frequency response changes very little with distance beyond about 5 units.

The phase versus frequency behavior of the five-source array is shown in Fig. 16, where responses at 45° off axis at distances of 20 and 100 000 units are plotted. The effects of linear phase lag and delay due to sample distance have been eliminated. The phase activity versus frequency is very similar to that of the two-source arrays, but is highly likely to be nonminimum phase due to the existence of the additional sources. The phase toggles rapidly between 0 and ± 180° as frequency increases.

4.3.3 Polar Peak-to-Peak Ripple versus Frequency

Fig. 17 exhibits a plot of polar peak-to-peak ripple, in decibels, versus frequency for the five-source equal-level array at a working distance of 20 units. Note that

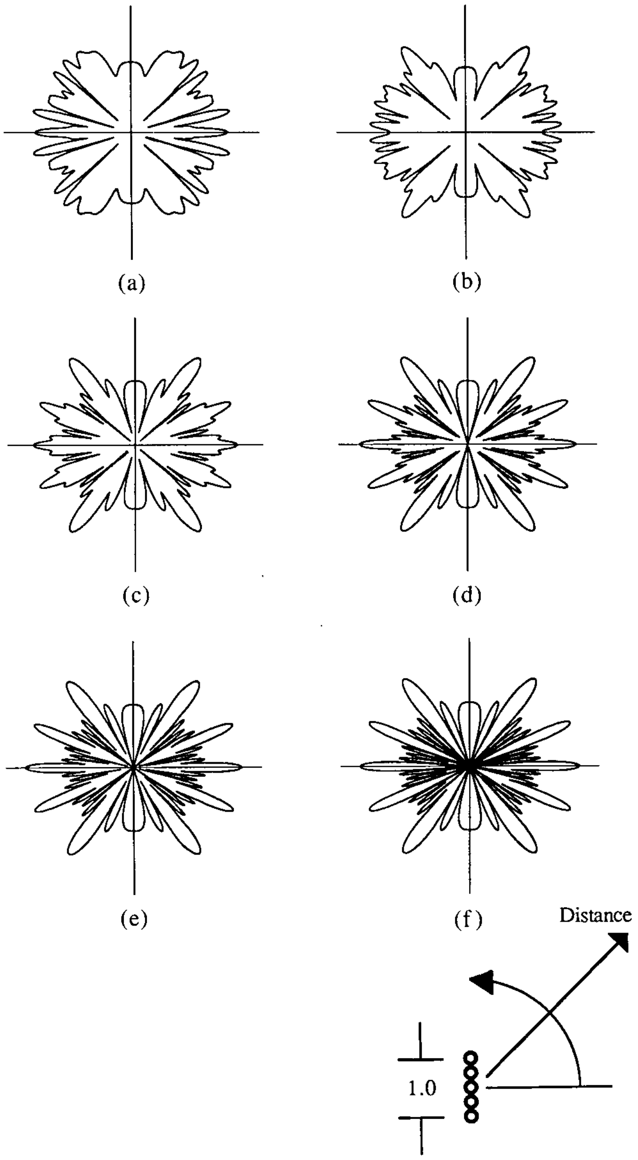


Fig. 12. Polar magnitude responses for five-source unit-length equal-level equal-polarity equal-spaced array at a fixed frequency of 10 Hz and different working distances. (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 20 units. (f) 100 000 units. Note that polar response changes very little with distance beyond roughly 10 units (10 array lengths).

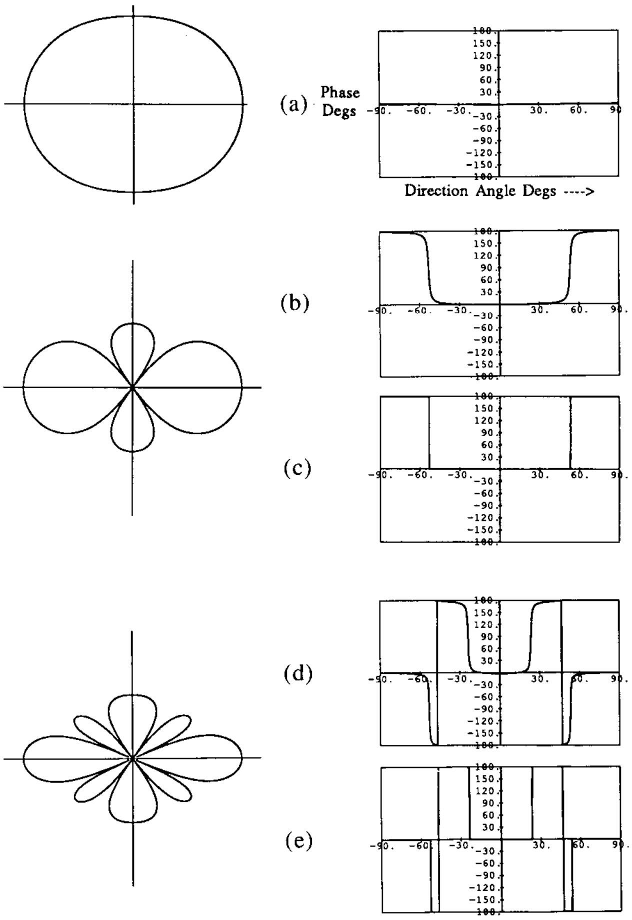


Fig. 13. Magnitude and phase polar responses at different frequencies and distances for five-source unit-length equal-level equal-polarity equal-spaced array. (a) f = 0.5 Hz, D = 20 units. (b) f = 1 Hz, D = 20 units. (c) f = 1 Hz, D = 100 000 units. (d) f = 2 Hz, D = 20 units. (e) f = 2 Hz, D = 100 000 units. Phase versus direction plots show phase of pressure at sample point versus off-axis direction. Effects of linear phase delay due to sample distance have been eliminated. Note that phase values switch between 0 and ± 180°, depending on the polar lobe on which the pressure sample point happens to be. Phase always starts out at 0° (on axis). At distances far from array, phase changes occur more abruptly with angle.

the ripple increases very rapidly above 0.35 Hz. The five-source array has somewhat better performance than the two-source 1.0-unit array, but significantly lower performance than the 0.25-unit two-source array (see Fig. 8). Additional data (not shown) indicate that the polar peak-to-peak ripple essentially does not change with working distances beyond about 10 units. This behavior is similar to that of the two-source arrays (see Fig. 9).

The following approximate equations relate the maximum operating frequency for omnidirectional radiation $f_{\max}$ to the array length for the five-source equal-level equal-polarity array:

$$
\begin{aligned}
f_{\max} &\approx 0.35\,\frac{c}{L}\,, \quad \text{for 3-dB peak-to-peak polar ripple} \\
&\approx 0.40\,\frac{c}{L}\,, \quad \text{for 4-dB peak-to-peak polar ripple} \\
&\approx 0.48\,\frac{c}{L}\,, \quad \text{for 6-dB peak-to-peak polar ripple} \\
&\approx 0.56\,\frac{c}{L}\,, \quad \text{for 3-dB peak-to-peak polar ripple}
\end{aligned}
\tag{11}
$$

where c is the velocity of sound and L the length of the array (center-to-center distance of outside sources). Compare these multipliers to the previous values for the two-source arrays given in Eqs. (9) and (10).

4.3.4 Discussion

At low frequencies, below about 0.35 Hz, the five-source array exhibits mostly omnidirectional behavior. The upper frequency limit for omnidirectional radiation occurs at the frequency where the length of the array is about one-third wavelength, which is somewhat higher than for the two-element array. Below this frequency, the efficiency is five times that of the single source, while the maximum output is 25 times higher.

The five-source equal-level array of 1.0-unit center-to-center length operates slightly higher in frequency than the two-source equal-level array with 1.0-unit center-to-center spacing, but significantly lower than the two-source equal-level array with 0.25-unit center-to-center spacing. The phase versus frequency curve is nonminimum phase. The phase versus angle and phase versus frequency curves alternate between 0 and ± 180°.

The directional characteristics essentially do not

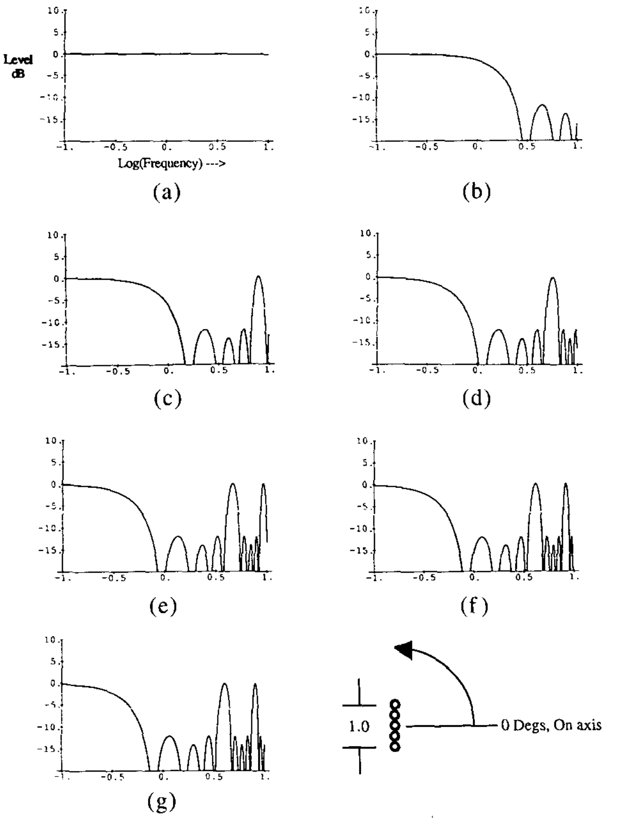


Fig. 14. Magnitude versus frequency responses for five-source unit-length equal-level equal-polarity equal-spaced array at a constant distance of 20 units and frequency range of 0.1 to 10 Hz. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). Responses are shown at angles ranging from 0 to +90° with steps of 15°. (a) 0°. (b) 15°. (c) 30°. (d) 45°. (e) 60°. (f) 75°. (g) 90°. Note that response progressively gets rougher as angle increases, similarly to two-source arrays.

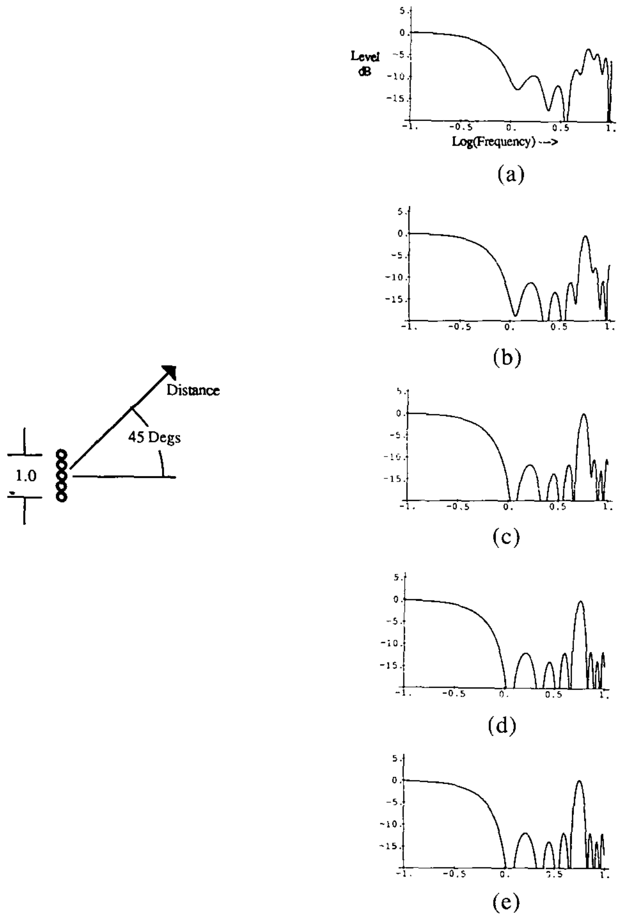


Fig. 15. Magnitude versus frequency responses at a fixed angle of 45° and different working distances for five-source unit-length equal-level equal-polarity equal-spaced array, with frequency range of 0.1 to 10 Hz. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 100 000 units. Note that frequency response changes very little with distance beyond about 5 units.

change beyond a point that is roughly 10 times the length of the array (roughly the same as for the two-source equal-level array).

4.4 Five-Source Bessel Array with Overall Center-to-Center Length of 1.0 Unit

The Bessel configuration is used to gain increased acoustic output without the severe narrowing directional characteristics with frequency exhibited by the equal-level equal-polarity equal-spaced line arrays. The Bessel array is said to have the same overall directional pattern as one of the sources that make up the array.

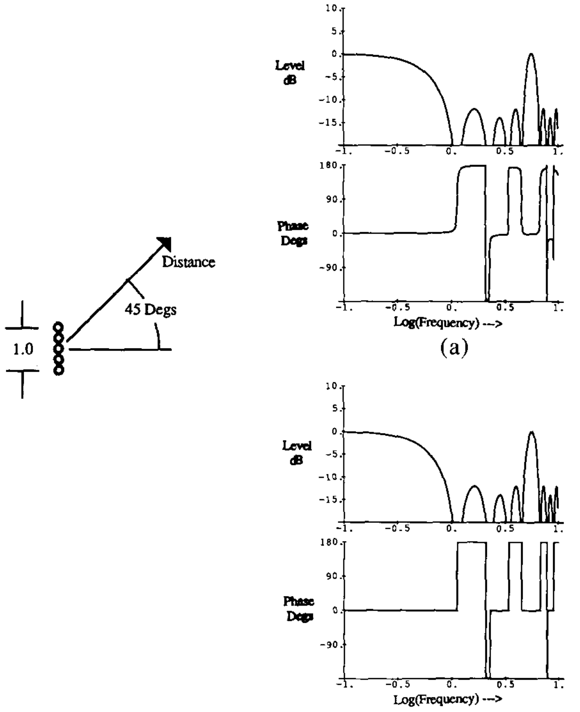


Fig. 16. Off-axis 45° magnitude and phase versus frequency plots over frequency range of 0.1 to 10 Hz for five-source unit-length equal-level equal-polarity equal-spaced array. (a) 20 units. (b) 100 000 units. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). Effects of linear phase lag and delay due to sample distance have been eliminated. Phase activity versus frequency is very similar to two-source arrays, but is highly likely to be nonminimum phase due to existence of additional sources. Phase toggles rapidly between 0 and ± 180° as frequency increases.

The following simulation uses omnidirectional point sources to form the Bessel structure. The degree to which the overall polar response matches an omnidirectional pattern is used to judge the effectiveness of the Bessel array. The five-source Bessel array contains the fewest number of sources of the three analyzed Bessel configurations. The characteristics and calculated parameters for the five-source Bessel array with 1.0-unit overall length are shown in Table 5.

Because of the much greater upper frequency of the Bessel array, all the bandwidth product values are much higher than those for the previous arrays. However, the efficiency is only about 14% (+0.6 dB) greater than that of a single source. With the higher power handling of 3.5 times a single source, the maximum acoustic output of the Bessel array is the same as that of the two-source arrays.

4.4.1 Polar Responses

The polar magnitude responses of the five-source Bessel array, at constant distance, are shown in Fig. 18. The polars are displayed at half-decade intervals from 0.316 to 100 Hz at a working distance of 20 units. The polar plot covers a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

Note the much greater high-frequency range of operation as compared with the previous arrays. The polar ripple does not get significant until frequencies higher than about 10 Hz.

The polar magnitude responses for the five-source Bessel array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 19. Polars at distances of 1.25, 2.5, 5, 10, 20, 40, and 100 000 units are plotted. Note that, unlike for the previous arrays, the polar ripple appears to get smaller and smaller

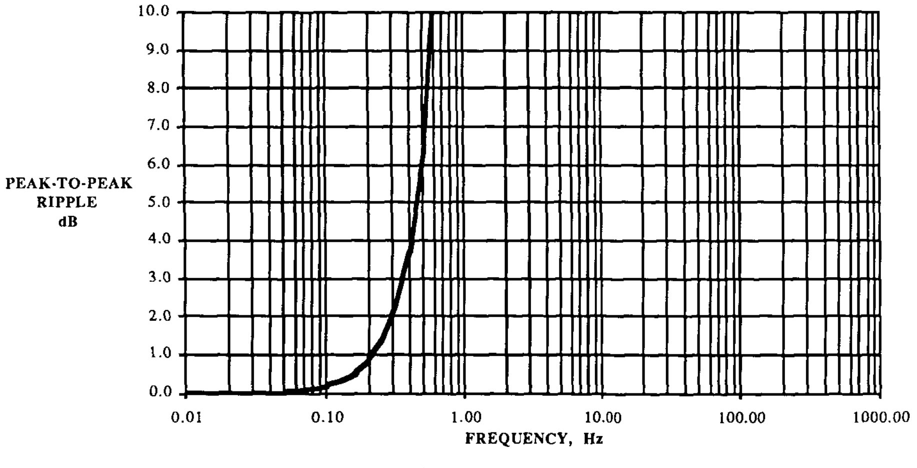


Fig. 17. Plot of polar peak-to-peak ripple versus frequency for five-source unit-length equal-level equal-polarity equal-spaced array at a working distance of 20 units. Note that ripple increases very rapidly above 0.35 Hz. Five-source unit-length array has somewhat better performance than two-source unit-length array, but significantly lower than 0.25-unit two-source array (see Fig. 8). Velocity of propagation 1 unit/s.

the farther away you get from the array. However, a limit of about 1.2 dB peak-to-peak ripple appears to exist even at the farthest distance. This figure is confirmed in [2, table 2].

For omnidirectional radiation, at a distance of 20 units, with no more than 6-dB peak-to-peak ripple, the five-source Bessel array is usable up to beyond 18 Hz. As compared to an equal-level equal-polarity two-source array with center-to-center spacing equal to the overall center-to-center spacing of the Bessel array (0.33 Hz from Table 3), this represents an increase in upper frequency of about 55 times (≈18/0.33).

A further study of the variation of polar ripple with distance was performed by simulating at the much higher frequency of 100 Hz (where the array length is 100 wavelengths) and then varying the working distance from 10 to 1000 units in three steps of one decade each. Fig. 20 shows the results of these simulations. The polar at 10-unit distance is unusable due to severe polar ripple (about 40 dB peak to peak). It settles down to about 2-dB peak-to-peak ripple at a distance of 1000 units (a long way away). It appears that there is no effective upper limit to the frequency of operation of the Bessel array if you can get far enough away. Practically, however, working distances in the range of 5 –20 times the length of the array will define the operation of the array.

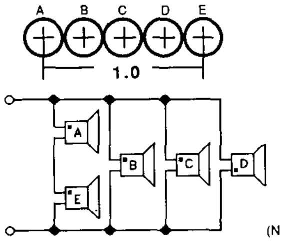
Fig. 21 shows a series of phase polar responses (phase versus direction angle) at a constant distance of 20 units and frequencies of 0, 0.1, 0.5, 1, 2, 4, 5, 10, and 20 Hz. The delay effects of the working distance have been compensated for, thus making the on-axis phase zero in every case. Also shown is a phase polar response at a distance of 100 000 units at 20 Hz. The phase curves exhibit a highly nonlinear sinusoidal-like variation of phase with angle with a peak-to-peak amplitude of ±90°. For a fixed angular increment, the number of oscillation cycles increases with frequency.

4.4.2 Frequency Responses

The magnitude versus frequency responses of the five-source Bessel array, at constant distance, are shown in Fig. 22. The responses cover the range from 0.1 to 10 Hz, and are plotted at angles ranging from 0 to +90°, with steps of 15°, at a working distance of 20 units. Unlike the previous equal-level arrays, the ripple does not increase continually with angle.

Table 5. Array type: five-source Bessel array.

Configuration:



(Note Polarity Dots!)

Number Units (N): 5
Overall Length (c-c): 1.0
Strengths: 0.5 : 1 : 1 : -1 : 0.5
Impedance (Z_{in}): 2/7 = 0.286 (-5.4 dB)
Voltage Sensitivity: 2 (+6.0 dB)
Efficiency(η_o): 8/7 = 1.143 (+0.6 dB)
Maximum Input Power (P_{in}): 7/2 = 3.5 (+5.4 dB)
Maximum Acoustic Output Power (P_{out}): 4 (+6.0 dB)
Maximum Sound Pressure Level: 2 (+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->		5	10	20
Ripple (dB):	3	2.05	4.00	8.00
	4	3.00	6.00	11.00
	6	4.50	8.80	18.00

Efficiency-Bandwidth Product (η_o x F_{max}):

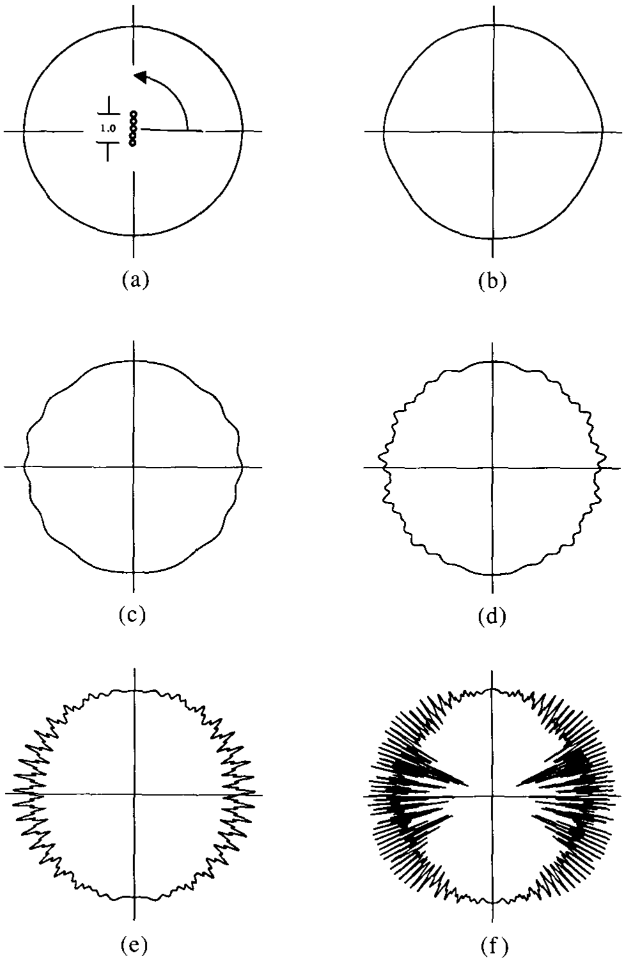
Distance ->		5	10	20
Ripple (dB):	3	2.34	4.57	9.14
	4	3.43	6.86	12.57
	6	5.14	10.06	20.57

Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->		5	10	20
Ripple (dB):	3	8.20	16.00	32.00
	4	12.00	24.00	44.00
	6	18.00	35.20	72.00

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->		5	10	20
Ripple (dB):	3	1.64	3.20	6.40
	4	2.40	4.80	8.80
	6	3.60	7.04	14.40



(a) (b)

(c) (d)

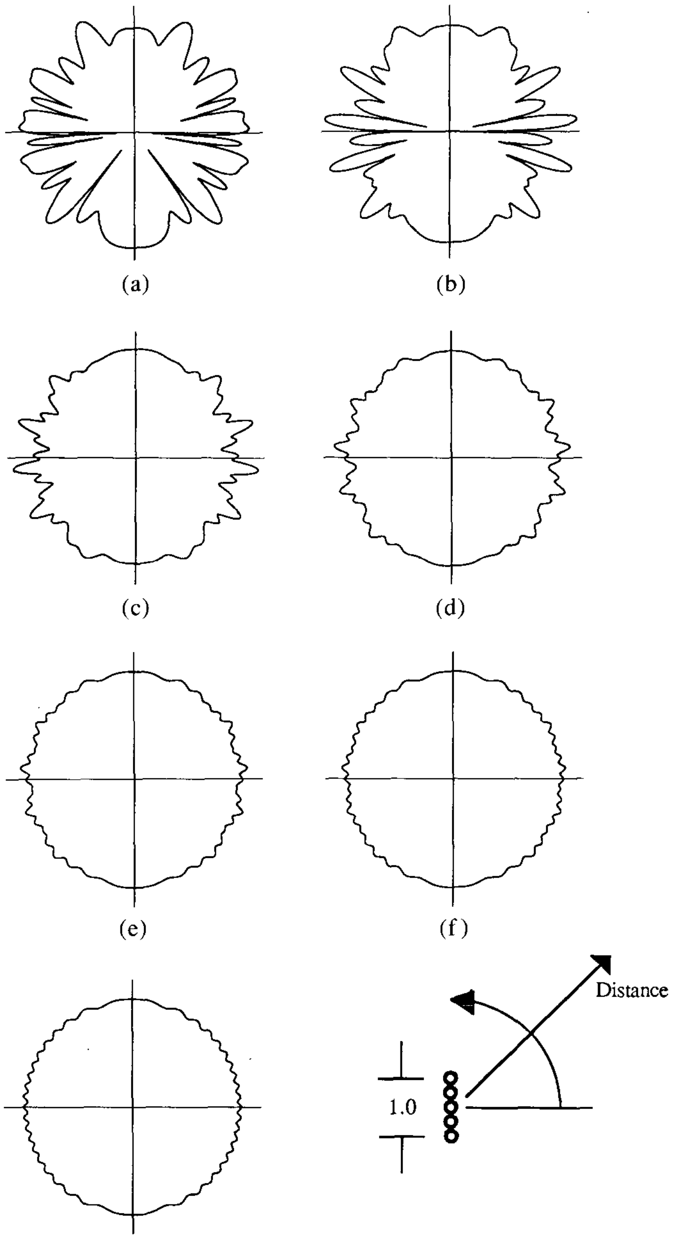
(e) (f)

Fig. 18. Polar magnitude responses for five-source unit-length Bessel array at a constant working distance of 20 units. Polars are displayed at half-decade intervals from 0.316 to 100 Hz. (a) 0.316 Hz. (b) 1 Hz. (c) 3.16 Hz. (d) 10 Hz. (e) 31.6 Hz. (f) 100 Hz. Note much greater high-frequency range of operation as compared with previous arrays. Polar ripple does not get significant until frequencies higher than about 10 Hz, where line length is 10 wavelengths. Polar plots cover a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that on-axis level is 0 dB.

The magnitude versus frequency responses at a fixed angle of 45° and at different working distances are shown in Fig. 23. Note the wider frequency range of 0.1 to 100 Hz. Responses at distances from 1.25 to 160 units with 1:2 steps are simulated, in addition to one at 100 000 units. Note, however, that, unlike for the previous arrays, the frequency response ripple decreases continually with distance until about a 2-dB peak-to-peak ripple is attained. This again reinforces the observation that the Bessel array performance can reach any arbitrary upper frequency if you move far enough away from the array.

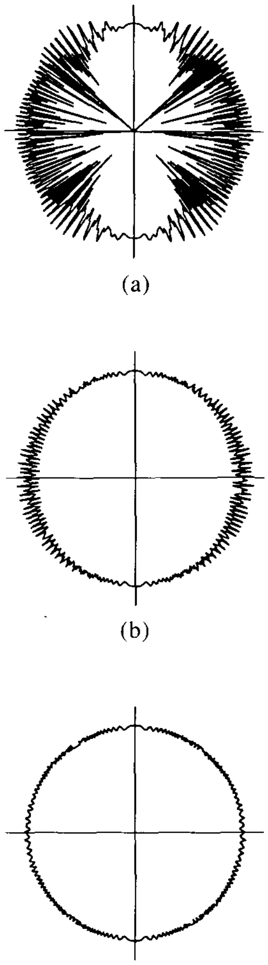
The phase versus frequency behavior of the five-source Bessel array is shown in Fig. 24, with magnitude, phase, and group delay responses at 45° off axis at distances of 20 units. Both log and linear frequency scale plots are shown, up to a frequency of 10 Hz. The phase varies nonlinearly, in a somewhat sinusoidal manner with frequency, oscillating between ±90°. The magnitude response is mostly flat, with peak-to-peak ripple, with more amplitude variations per unit frequency.

Because the magnitude is mostly flat and the phase varies dramatically with frequency, this magnitude–phase behavior versus frequency is highly nonlinear and nonminimum phase. The group delay plots of Fig. 24(c) and (f) indicate an effective oscillatory peak shift of the acoustic position of about ±25% the length of the array as the frequency is increased. I am not going to venture an opinion on whether or not this is audible.



(a) (b) (c) (d) (e) (f) (g)

Fig. 19. Polar magnitude responses for five-source unit-length Bessel array at fixed frequency of 10 Hz and different working distances. (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 20 units. (f) 40 units. (g) 100 000 units. Note that, unlike previous arrays, polar ripple appears to get smaller and smaller the farther away you get from the array. Polar plots cover a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that on-axis level is 0 dB.



(a) (b) (c)

Fig. 20. Polar magnitude responses for five-source unit-length Bessel array, but at a much higher fixed frequency of 100 Hz and much farther working distances, covering the range of 10 to 1000 units in three steps of one decade each. (a) 10 units. (b) 100 units. (c) 1000 units. Note that even at this high frequency, where line is 100 wavelengths long, at large distances polar peak-to-peak ripple settles down to relatively small values. Polar plots cover a range of 40 dB with +6 dB at the outer edge and −34 dB at the center. All polars are normalized so that on-axis level is 0 dB.

4.4.3 Polar Peak-to-Peak Ripple versus Frequency

Fig. 25 exhibits a plot of polar peak-to-peak ripple, in decibels, versus frequency for the five-source Bessel array at a working distance of 20 units. Observe that the ripple increases much more gradually with increasing frequency as compared to the equal-level arrays. Note also the much extended bandwidth of operation as compared to the previous arrays. Also observe the plateau in the curve between 0.5 and 1.1 Hz, where the ripple is about 1.3 dB. At a peak-to-peak ripple of 6 dB, operation extends up to a frequency of 18 Hz (line length of 18 wavelengths).

To investigate the behavior of ripple with increasing distance, numerous plots of ripple versus frequency were done over the distance range of 2.5 to 100 000 units. These data are plotted in Fig. 26. It is quite evident that the operation of the Bessel array improves in direct proportion to the working distance away from the array. Note that at points very far from the array, the peak-to-peak ripple attains a constant value of about 1.3 dB; this is the source of the plateau noted previously.

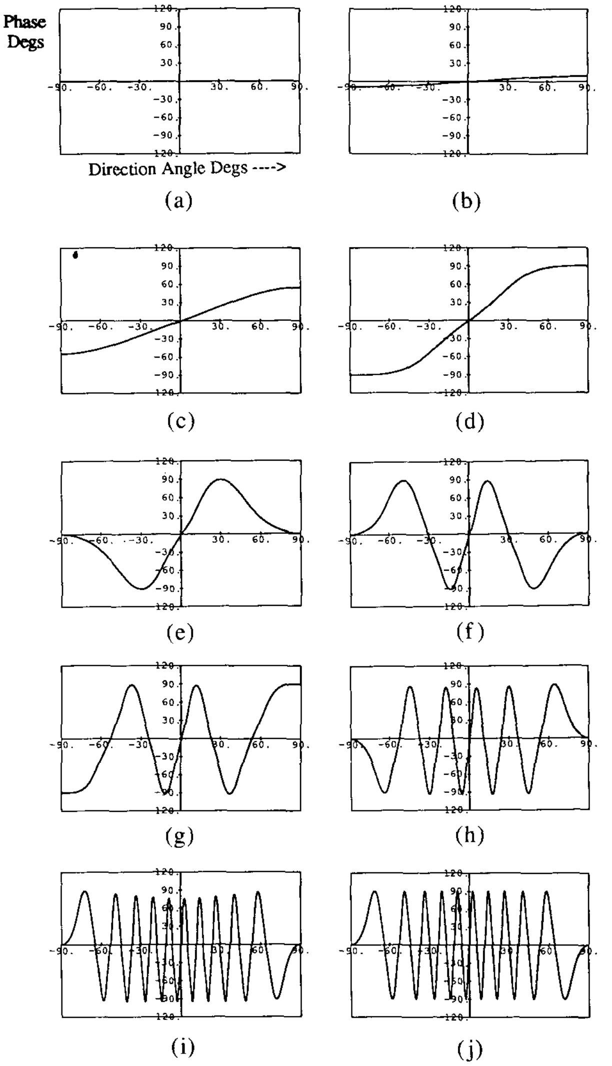


Fig. 21. Phase polar responses (phase versus direction angle) for five-source unit-length Bessel array at constant distance of 20 units and different frequencies. (a) 0 Hz. (b) 0.1 Hz. (c) 0.5 Hz. (d) 1 Hz. (e) 2 Hz. (f) 4 Hz. (g) 5 Hz. (h) 10 Hz. (i) 20 Hz. (j) 20 Hz at a distance of 100 000 units. Delay effects of working distance have been compensated for, thus making on-axis phase zero in every case. Phase curves exhibit highly nonlinear sinusoidal-like variation of phase with angle with a peak-to-peak amplitude of ±90°. For a fixed angular increment, number of oscillation cycles increases with frequency.

Fig. 27 shows a plot of maximum operating frequency versus operating distance for the five-source Bessel array. Contours of equal peak-to-peak ripple at values of 3, 6, and 9 dB are plotted. The direct relationship between maximum frequency and operating distance is evident. The contours of constant peak-to-peak ripple form straight lines on the graph, except for slight deviations at small distances.

The following approximate equations relate the maximum operating frequency for omnidirectional radiation $f_{\max}$ to array length L and operating distance D, for the five-element Bessel array. Note the dependence on distance, which was absent in the previous array equations [Eqs. (9)–(11)].

$$\begin{aligned} f_{\max} &\approx 0.40\,\frac{c}{L}\,D, && \text{for 3-dB peak-to-peak polar ripple} \\ &\approx 0.55\,\frac{c}{L}\,D, && \text{for 4-dB peak-to-peak polar ripple} \\ &\approx 0.90\,\frac{c}{L}\,D, && \text{for 6-dB peak-to-peak polar ripple} \\ &\approx 1.4\,\frac{c}{L}\,D, && \text{for 9-dB peak-to-peak polar ripple} \end{aligned} \qquad (12)$$

where

D = normalized operating distance, = d/L
c = velocity of sound
d = working distance away from center of array
L = length of array (center-to-center distance of outside sources).

For comparison with the equations for the previous arrays [Eqs. (9)–(11)], the following equations evaluate Eqs. (12) at a distance of 20 units:

$$\begin{aligned} f_{\max} &\approx 8\,\frac{c}{L}, && \text{for 3-dB peak-to-peak polar ripple} \\ &\approx 11\,\frac{c}{L}, && \text{for 4-dB peak-to-peak polar ripple} \\ &\approx 18\,\frac{c}{L}, && \text{for 6-dB peak-to-peak polar ripple} \\ &\approx 28\,\frac{c}{L}, && \text{for 9-dB peak-to-peak polar ripple} \end{aligned} \qquad (13)$$

Note the large multipliers as compared to the previous array equations.

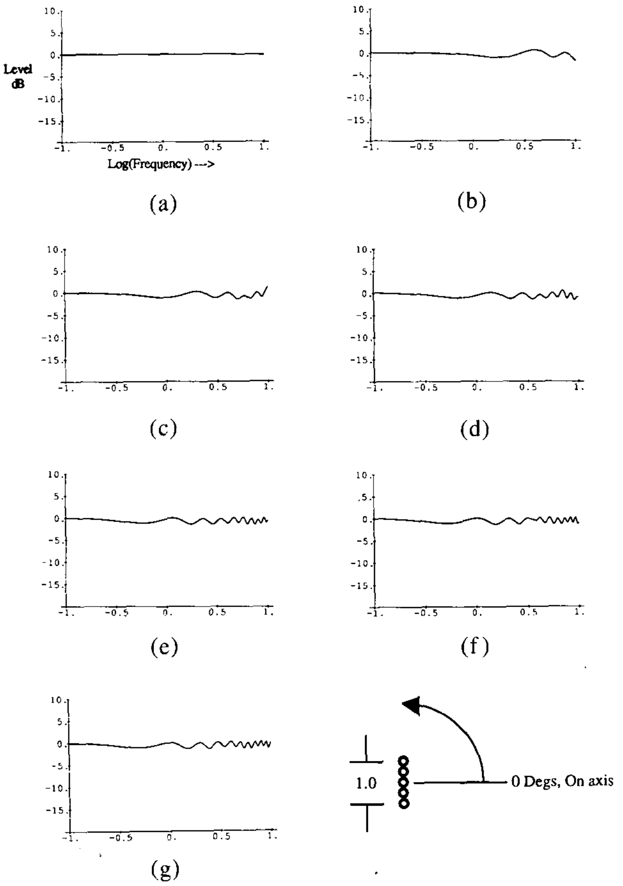
4.4.4 Discussion

The five-element Bessel array provides a very impressive increase in the bandwidth of operation when compared to equivalent two- and five-source equal-level equal-polarity equal-spaced arrays. The efficiency–bandwidth product, power–bandwidth product, and power–bandwidth product per unit are all very high in comparison to the previous arrays.

When compared to a two-source equal-level in-phase array, a five-source Bessel array is 2.4 dB less efficient, can handle 1.75 (+2.4 dB) more power, has the same maximum midband acoustic output power, and is usable for omnidirectional radiation 10 times higher in frequency. A working distance of 20 times the length of the Bessel array is assumed, with the length of the Bessel array (center-to-center measurement) being four times that of the two-source array.

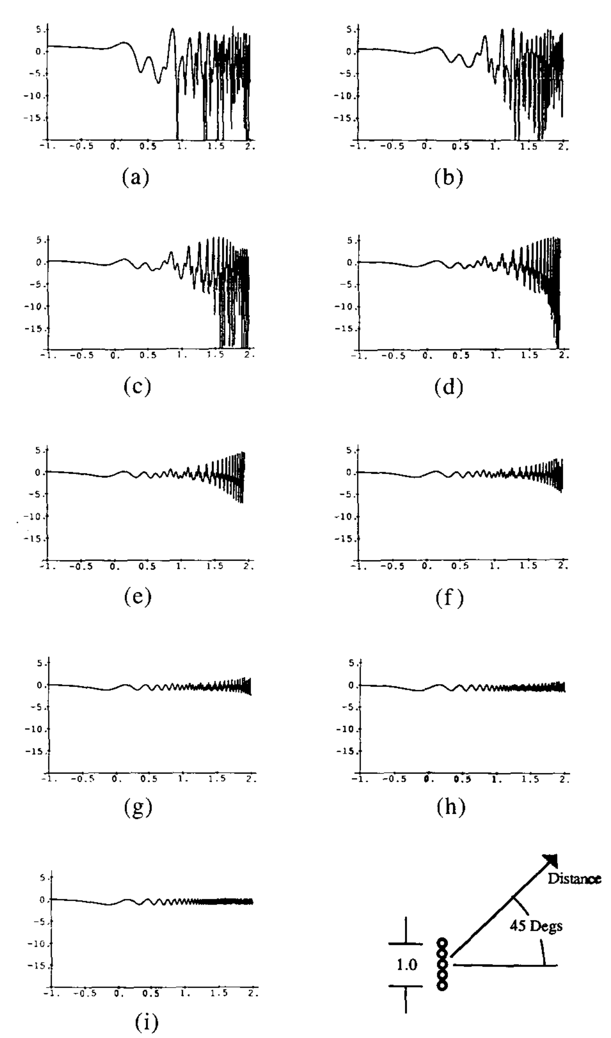
The very nonlinear phase behavior with direction angle and frequency appears to be the single major problem with the Bessel array. Whereas a single point source has true omnidirectional radiation, it does not exhibit any variation of phase with angle or frequency (neglecting transport delay between source and sample point). The Bessel array's off-axis variation of phase with angle and frequency makes it very difficult to use it with any other sources. Computation of group delay versus frequency at an angle of 45° indicates an oscillatory movement of the acoustic position about ±25% of the array's length.

Eqs. (12) clearly show that the high-frequency limit of the Bessel array increases in direct proportion to the working distance from the array. This is in contrast to the behavior of the two- and five-source equal-level equal-polarity equal-spaced arrays, where the performance does not change beyond a fairly close distance measured in terms of the array length (about 10 times the array length). This means that the Bessel array is not like a conventional source that exhibits a typical near-field/far-field difference in its behavior. The Bessel array does not have a definite near-field/far-field boundary which defines its behavior.



(a) (b) (c) (d) (e) (f) (g)

Fig. 22. Magnitude versus frequency responses for five-source unit-length Bessel array at a constant working distance of 20 units. Responses cover the range from 0.1 to 10 Hz and are shown at angles ranging from 0 to +90°, with steps of 15°. (a) 0°. (b) 15°. (c) 30°. (d) 45°. (e) 60°. (f) 75°. (g) 90°. Unlike previous equal-level arrays, ripple does not increase continually with angle. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = Hz, etc.).



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 23. Magnitude versus frequency responses for five-source unit-length Bessel array at a fixed angle of 45° and working distances of 1.25 to 160 units with 1:2 steps and at 100 000 units. Note wider frequency range of 0.1 to 100 Hz. (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 20 units. (f) 40 units. (g) 80 units. (h) 160 units. (i) 100 000 units. Unlike previous arrays frequency response ripple decreases continually with distance until about a 2-dB peak-to-peak ripple is attained. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.).

4.5 Seven-Source Bessel Array with Overall Center-to-Center Length of 1.5 Units

As noted in Sec. 1, the seven-source Bessel array actually has six sources instead of seven, because the middle source has a drive level of zero, and thus does not have to be there physically. The space for the removed source must exist to preserve proper operation of the array, however. The length of the seven-source Bessel array was chosen to be 1.5 units (center-to-center spacing of outside sources). This specific length was selected because it is the length of the seven-source array when composed of the same-size units as the five-source array. The characteristics and calculated parameters for the seven-source Bessel array are shown in Table 6.

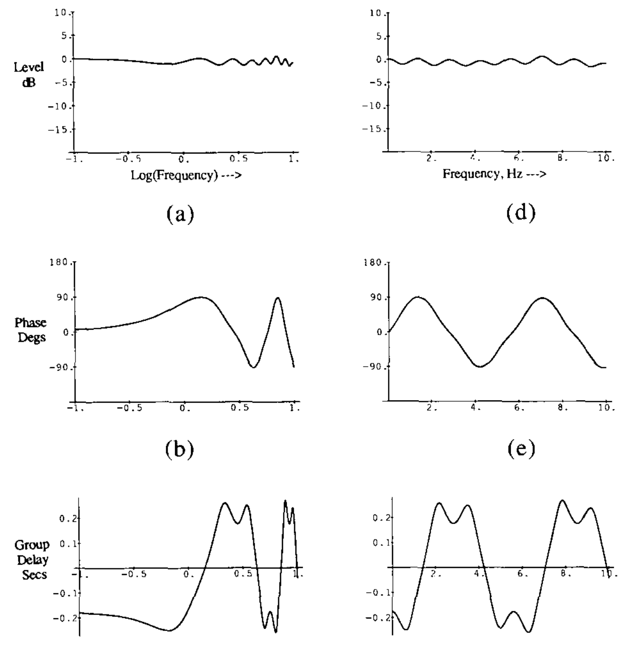


Fig. 24. Magnitude, phase, and group delay versus frequency responses for five-source unit-length Bessel array at 45° off axis and a working distance of 20 units. Both logarithmic and linear frequency scale plots are shown, up to a frequency of 10 Hz. (a) Magnitude, log scale. (b) Phase, log scale. (c) Group delay, log scale. (d) Magnitude, linear scale. (e) Phase, linear scale. (f) Group delay, linear scale. Magnitude response is mostly flat, with about a 2-dB peak-to-peak ripple. Phase varies nonlinearly in a somewhat sinusoidal manner with frequency, oscillating between ± 90° which indicates a nonminimum phase response. Group delay plot indicates an effective oscillatory peak shift of acoustic position of about ± 25% the length of the array, as the frequency is increased.

The efficiency of the seven-source Bessel array is actually about 11% (0.5 dB) less than that of one of the single sources that make up the array. The efficiency is also about 22% less than that of the five-source Bessel array. With the increased power handling of 4.5 (+6.5 dB), this generates a maximum output of 4 W (+6 dB), which is the same as the maximum output of the two-source equal-level equal-polarity array and the five-source Bessel array.

Because of the additional element required and the lower bandwidth, this array's power–bandwidth product per unit is less than half that of the five-source Bessel array. For this reason, very few response curves were generated for the seven-source Bessel array because of its relatively poor characteristics.

4.5.1 Polar Responses

Only one polar response was generated for the seven-source Bessel array. This is shown in Fig. 30 (Sec. 5.3), which compares the polars of all the arrays at a specific frequency and working distance.

4.5.2 Frequency Responses

Only one frequency response was calculated for the seven-source Bessel array and is shown in Fig. 31 (Sec. 5.4), where the response is compared with those of the other analyzed arrays.

4.5.3 Polar Peak-to-Peak Ripple versus Frequency

Fig. 28 exhibits a plot of the seven-source Bessel array's polar peak-to-peak ripple, in decibels, versus

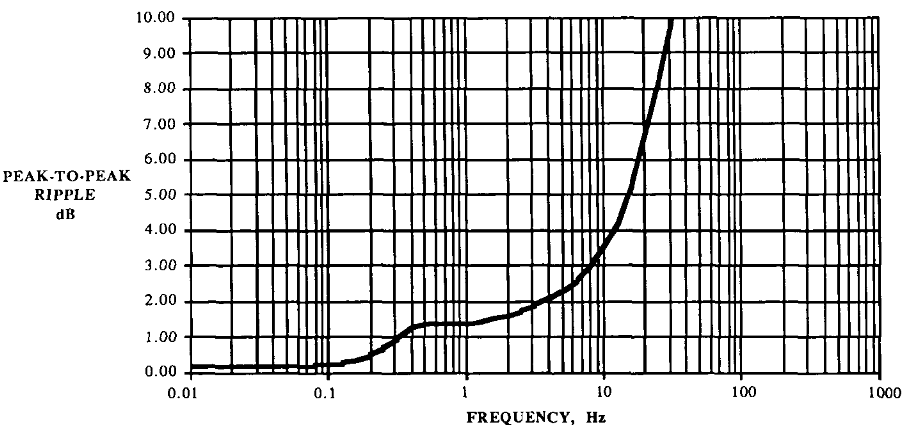


Fig. 25. Polar magnitude peak-to-peak ripple versus frequency for five-source unit-length Bessel array at a working distance of 20 units. Observe that ripple increases much more gradually with increasing frequency as compared to equal-level arrays. Note also much extended bandwidth of operation as compared to previous arrays. Velocity of propagation 1 unit/s.

frequency for working distances of 5, 10, 20, 100, and 100 000 units. At large distances, the ripple attains a minimum plateau value of about 1.0 dB. Close to the array ($D = 5$), the ripple does not go below about 2 dB. In general, the curves are shifted to the left, as compared to the five-source Bessel array, which indicates lower frequencies of operation.

4.5.4 Discussion

As stated in the introduction to this section, the disappointing performance of the seven-source Bessel array as compared to the five-source Bessel array makes it undesirable for practical use.

4.6 Nine-Source Bessel Array with Overall Center-to-Center Length of 2.0 Units

As noted in Sec. 1, the nine-source Bessel array actually has seven sources instead of nine, because two of the sources have drive levels of zero, and thus do not have to be in the array. The spaces for the removed sources must exist to preserve proper operation of the array, however. The overall length of the nine-source Bessel array is 2.0 units (center-to-center spacing of outside sources). This length was chosen because it results from using the same size sources as those used in the previous five-source arrays. The characteristics and the calculated parameters for the nine-source Bessel array are shown in Table 7.

The efficiency of the nine-source Bessel array is actually about 27% (1.4 dB) less than that of one of the single sources that make up the array. The efficiency is also about 36% less than that of the five-source Bessel array. With the increased power handling of 5.5 (+7.4 dB), this generates a maximum output of 4 W (+6 dB), which is the same as the maximum output of the two-source equal-level equal-polarity array and the five- and seven-source Bessel arrays. Also, the maximum upper frequency of the nine-source Bessel array (assuming 4-dB peak-to-peak ripple) is less than one-eighth that of the five-source Bessel array.

Because of the two additional elements required and drastically lower bandwidth, this array's power–bandwidth product per unit is less than one-tenth that of the five-source Bessel array. This very

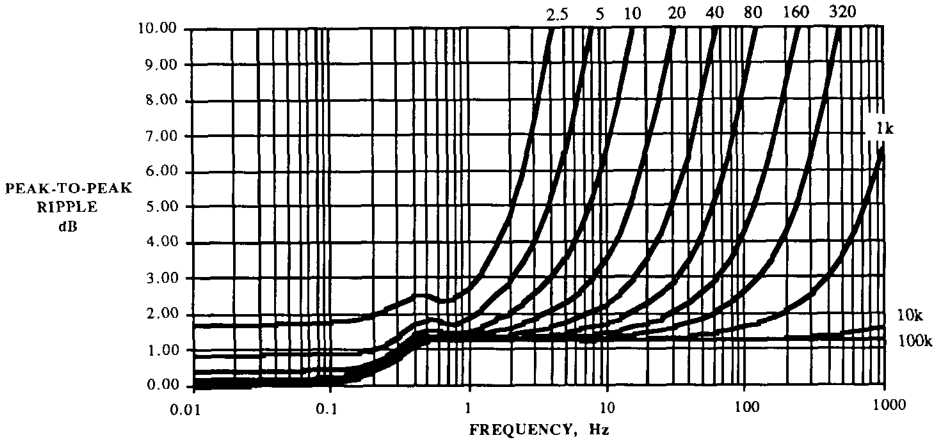


Fig. 26. Polar magnitude peak-to-peak ripple versus frequency five-source unit-length Bessel array at working distances of 2.5 to 100 000 units. It is quite evident that operation of the Bessel array improves in direct proportion to the working distance away from the array. Velocity of propagation 1 unit/s.

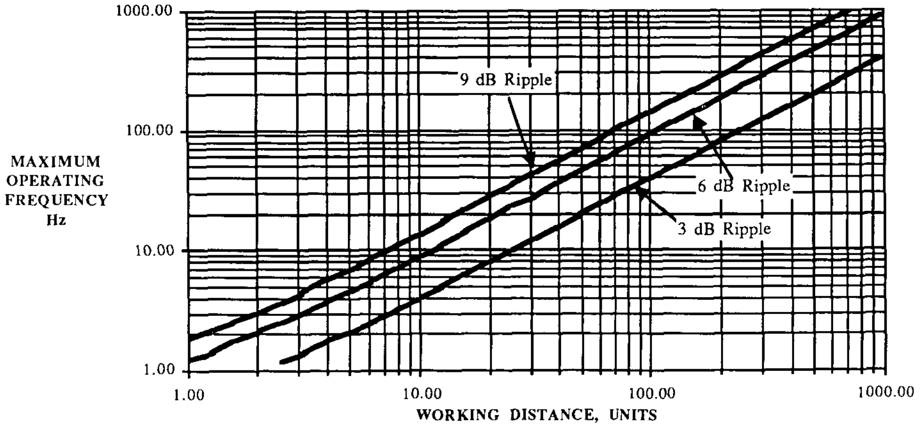


Fig. 27. Plot of maximum operating frequency versus working distance for five-source unit-length Bessel array. Contours of equal peak-to-peak ripple at values of 3, 6, and 9 dB are plotted. Direct relationship between maximum frequency and operating distance is clearly shown. Velocity of propagation 1 unit/s.

poor performance takes it out of the running for any practical application. For this reason, very few response curves were generated for the nine-source Bessel array.

4.6.1 Polar Responses

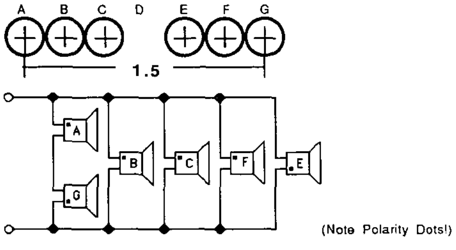
Only one polar response was generated for the nine-source Bessel array. This is shown in Fig. 30 (Sec. 5.3), which compares the polars of all the arrays at a specific frequency and working distance.

4.6.2 Frequency Responses

Only one frequency response was calculated for the nine-source Bessel array and is shown in Fig. 31 (Sec. 5.4), where the response is compared with those of the other analyzed arrays.

Table 6. Array type: seven- (six)-source Bessel array.

Configuration:



Number Units (N):	6	
Overall Length (c-c):	1.5	
Strengths:	0.5 : 1 : 1 : 0 : -1 : 1 : -0.5	
Impedance (Z_{in}):	2/9 = 0.222	(-6.5 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	8/9 = 0.889	(-0.5 dB)
Maximum Input Power (P_{in}):	9/2 = 4.5	(+6.5 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	1.36	2.20	4.00
4	1.73	2.95	5.55
6	2.43	4.32	8.30

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	1.21	1.96	3.56
4	1.54	2.62	4.93
6	2.16	3.84	7.38

Power-Bandwidth Product (P_{out} x F_{max}):

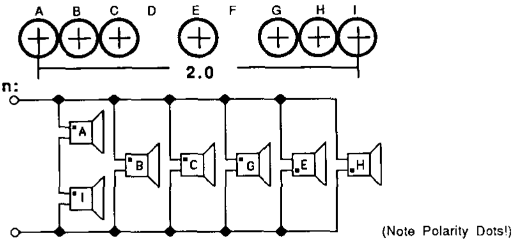
Distance ->	5	10	20
Ripple (dB): 3	5.44	8.80	16.00
4	6.92	11.80	22.20
6	9.72	17.28	33.20

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	0.91	1.47	2.67
4	1.15	1.97	3.70
6	1.62	2.88	5.53

Table 7. Array type: nine- (seven)-source Bessel array.

Configuration:



Number Units (N):	7	
Overall Length (c-c):	2.0	
Strengths:	0.5 : 1 : 1 : 0 : -1 : 0 : 1 : -1 : 0.5	
Impedance (Z_{in}):	2/11 = 0.182	(-7.4 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	8/11 = 0.727	(-1.4 dB)
Maximum Input Power (P_{in}):	11/2 = 5.5	(+7.4 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.13	0.46	0.78
4	0.46	1.06	1.30
6	1.50	1.96	3.95

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.09	0.33	0.56
4	0.33	0.77	0.95
6	1.09	1.42	2.87

Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.52	1.84	3.12
4	1.84	4.24	5.20
6	6.00	7.84	15.80

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	0.07	0.26	0.44
4	0.26	0.61	0.74
6	0.86	1.12	2.26

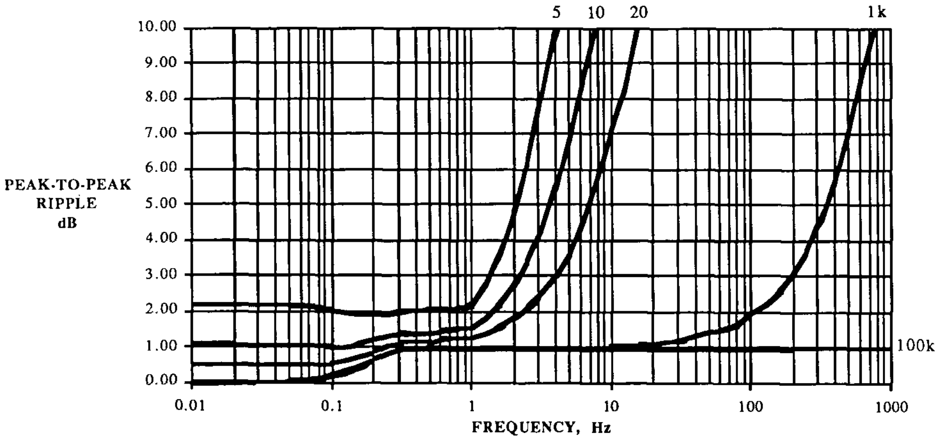


Fig. 28. Polar magnitude peak-to-peak ripple versus frequency for seven-source Bessel array of 1.5-unit length, at working distances of 5 to 100 000 units. At large distances, ripple attains a minimum plateau value of about 1.0 dB. In general, curves are shifted left as compared to five-source Bessel array, which indicates lower frequencies of operation.

4.6.3 Polar Peak-to-Peak Ripple versus Frequency

Fig. 29 shows a plot of the nine-source Bessel array's polar peak-to-peak ripple, in decibels, versus frequency for working distances of 5, 10, 20, 1000 and 100 000 units. At large distances, the ripple attains a minimum plateau value of about 3.6 dB, which is significantly higher than those of the previous Bessel arrays. As noted for the seven-source Bessel array, the curves are shifted even more to the left, as compared to the five-source Bessel array, which indicates an even lower bandwidth of operation.

4.6.4 Discussion

The performance of the nine-source Bessel array is significantly worse than even that of the seven-source Bessel array, which was previously judged undesirable for practical use. Its much lower efficiency, requirement of two more sources, and very much lower bandwidth definitely take it out of the running.

5 ARRAY COMPARATIVE ANALYSIS

A comparative analysis was done on all the analyzed arrays. This includes a master comparison table where all the array performance factors are shown, a series of performance ranking tables, a comparative display of polar responses and frequency responses, and a graph showing polar ripple versus frequency for all the arrays.

5.1 Tabular Comparison

Table 8 is a master tabular comparison of all the analyzed arrays, assuming a working distance of 20 units and a peak-to-peak polar ripple of 4 dB. The last four rows of the table indicate the clear superiority of

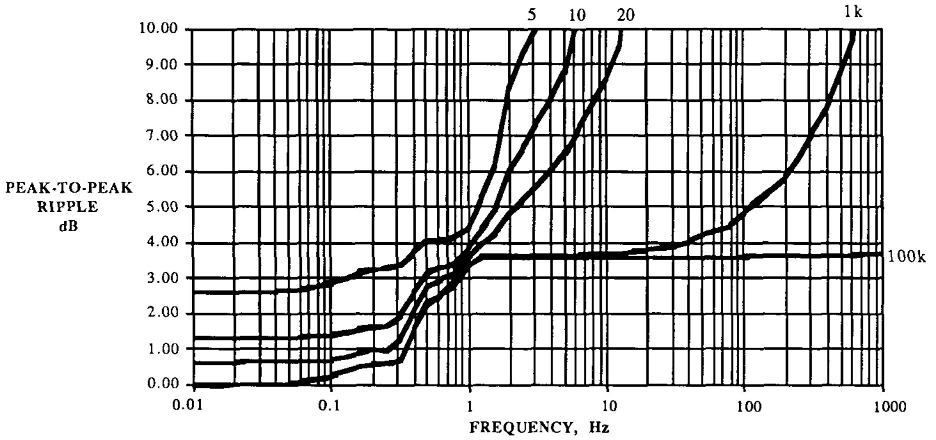


Fig. 29. Polar magnitude peak-to-peak ripple versus frequency for nine-source 2-unit-length Bessel array at working distances of 5 to 100 000 units. At large distances, ripple attains a minimum plateau value of about 3.6 dB, which is significantly higher than that of previous Bessel arrays. As noted for the seven-source Bessel array, curves are shifted even more left as compared to five-source Bessel array, which indicates an even lower bandwidth of operation.

Table 8. Comparison of array types.

ARRAY TYPE =	Single Source	2 Sources Equal Level (L=0.25)	2 Sources Equal Level (L=1.0)	5 Sources Equal Level & Spacing	5 Source Bessel	7(6) Source Bessel	9(7) Source Bessel
Configuration (to scale) =	o	oo	o o	ooooo	ooooo	ooo ooo	ooo o ooo
Number Units =	1	2	2	5	5	6	7
Overall Length (c-c) =	0	0.25	1.0	1.0	1.0	1.5	2.0
Impedance =	1.000	0.500	0.500	0.200	0.286	0.222	0.182
Voltage Sensitivity =	1	2	2	5	2	2	2
Efficiency =	1.000	2.000	2.000	5.000	1.143	0.889	0.727
Maximum Input Power =	1.0	2.0	2.0	5.0	3.5	4.5	5.5
Max. Output Power =	1.0	4.0	4.0	25.0	4.0	4.0	4.0
Maximum Sound Pressure Level =	1.0 (0 dB)	2.0 (+6 dB)	2.0 (+6 dB)	5.0 (+14 dB)	2.0 (+6 dB)	2.0 (+6 dB)	2.0 (+6 dB)
Maximum Upper Frequency = (Distance = 20, P-P Ripple = 4 dB)	Infinity	1.10	0.28	0.40	11.00	5.55	1.30
Efficiency-Bandwidth Product = (Distance = 20, P-P Ripple = 4 dB)	Infinity	2.20	0.56	2.00	12.57	4.93	0.95
Power Bandwidth Product = (Distance = 20, P-P Ripple = 4 dB)	Infinity	4.4	1.1	10.0	44.0	22.2	5.2
Power-Bandwidth Product per Unit = (Distance = 20, P-P Ripple = 4 dB)	Infinity	2.20	0.56	2.00	8.80	3.70	0.74

the five-source Bessel array as compared to the other analyzed arrays. The much higher bandwidth of operation is reflected in the high values of all the bandwidth products.

5.2 Performance Rankings

This section displays rankings for each of the analyzed arrays, for all the major array characteristics.

5.2.1 *Efficiency*

Table 9 shows the comparative rankings of the analyzed arrays for efficiency. As expected, the five-source equal-level equal-polarity equal-spaced array is at the top of the list. However, its high efficiency is mostly offset by its lower bandwidth of operation. The nine-source Bessel array is at the bottom of the list (27% less efficiency than a single source.).

5.2.2 *Power Handling*

The comparative rankings for input power handling are shown in Table 10. The nine-source Bessel array is at the top of this list. This is fortunate because it also has the lowest efficiency (Table 9). It would make a good heater.

5.2.3 *Maximum Acoustic Output Power*

Table 11 displays the ranking order for the array's maximum acoustic output power. The five-source equal-level array is seen to head the list. Even though this array provides high acoustic output power, its high frequency capabilities are limited. As can be seen, most of the analyzed arrays have maximum outputs of four times a single unit.

Table 9. Ranking for efficiency.

RANK	VALUE	ARRAY TYPE
1	5.00	5 Sources, Equal Level and Spacing
2	2.00	2 Sources, Equal Level (L=0.25)
3	2.00	2 Sources, Equal Level (L=1.0)
4	1.14	5 Source Bessel
5	1.00	Single Source
6	0.89	7(6) Source Bessel
7	0.73	9(7) Source Bessel

Table 10. Ranking for maximum input power.

RANK	VALUE	ARRAY TYPE
1	5.5	9(7) Source Bessel
2	5.0	5 Sources, Equal Level and Spacing
3	4.5	7(6) Source Bessel
4	3.5	5 Source Bessel
5	2.0	2 Sources, Equal Level (L=0.25)
6	2.0	2 Sources, Equal Level (L=1.0)
7	1.0	Single Source

Table 11. Ranking for maximum output power.

RANK	VALUE	ARRAY TYPE
1	25	5 Sources, Equal Level and Spacing
2	4	2 Sources, Equal Level (L=0.25)
3	4	2 Sources, Equal Level (L=1.0)
4	4	5 Source Bessel
5	4	7(6) Source Bessel
6	4	9(7) Source Bessel
7	1	Single Source

5.2.4 *Maximum Operating Frequency*

Table 12 ranks all the analyzed arrays for maximum operating frequency. Excluding the single source, the five-source Bessel array is seen to head the list with a large two-to-one margin. The widely separated two-source array is at the bottom of the list.

5.2.5 *Efficiency–Bandwidth Product*

The rankings for the efficiency–bandwidth product are shown in Table 13. Again, after excluding the single source, the five-source Bessel array's superiority is clearly shown, with a margin of greater than 2.5 over the second-place entry. The nine-source Bessel array is in next to last place.

5.2.6 *Power–Bandwidth Product*

Table 14 shows the rankings for the power–bandwidth product. The five-source Bessel array again heads the list, after excluding the single source. The seven- and nine-source Bessel arrays do a bit better in this comparison. The wide-separation two-source array is in last place.

5.2.7 *Power–Bandwidth Product per Unit*

Table 15 lists the rankings for the power–bandwidth product per unit. This parameter is a good figure of merit for comparing the arrays in that it shows how good the performance is on a per-unit basis. The five-source Bessel array is again on top, with the exception

Table 12. Ranking for maximum operating frequency. Distance = 20 units, peak-to-peak ripple = 4 dB.

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	11.00	5 Source Bessel
3	5.55	7(6) Source Bessel
4	1.30	9(7) Source Bessel
5	1.10	2 Sources, Equal Level (L=0.25)
6	0.40	5 Sources, Equal Level and Spacing
7	0.28	2 Sources, Equal Level (L=1.0)

Table 13. Ranking for efficiency–bandwidth product. Distance = 20 units, peak-to-peak ripple = 4 dB.

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	12.57	5 Source Bessel
3	4.93	7(6) Source Bessel
4	2.20	2 Sources, Equal Level (L=0.25)
5	2.00	5 Sources, Equal Level and Spacing
6	0.95	9(7) Source Bessel
7	0.56	2 Sources, Equal Level (L=1.0)

Table 14. Ranking for power–bandwidth product. Distance = 20 units, peak-to-peak ripple = 4 dB.

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	44.0	5 Source Bessel
3	22.2	7(6) Source Bessel
4	10.0	5 Sources, Equal Level and Spacing
5	5.2	9(7) Source Bessel
6	4.4	2 Sources, Equal Level (L=0.25)
7	1.1	2 Sources, Equal Level (L=1.0)

of the single source. The seven-source Bessel array is in a fairly strong second-place position. The nine-source Bessel array is in next to last place with a power-bandwidth product of about one-twelfth that of the five-source Bessel array.

5.3 Polar Response Comparison

Fig. 30 shows a comparison of the polars for all the analyzed arrays. All the polars were run at the same frequency (10 Hz) and working distance (20 units). The peak-to-peak polar ripple is listed on each plot. The superiority of the five-source Bessel array [Fig. 30(d)] is clearly evident.

Table 15. Ranking for power–bandwidth product per unit. Distance = 20 units, peak-to-peak ripple = 4 dB.

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	8.80	5 Source Bessel
3	3.70	7(6) Source Bessel
4	2.20	2 Sources, Equal Level (L=0.25)
5	2.00	5 Sources, Equal Level and Spacing
6	0.74	9(7) Source Bessel
7	0.56	2 Sources, Equal Level (L=1.0)

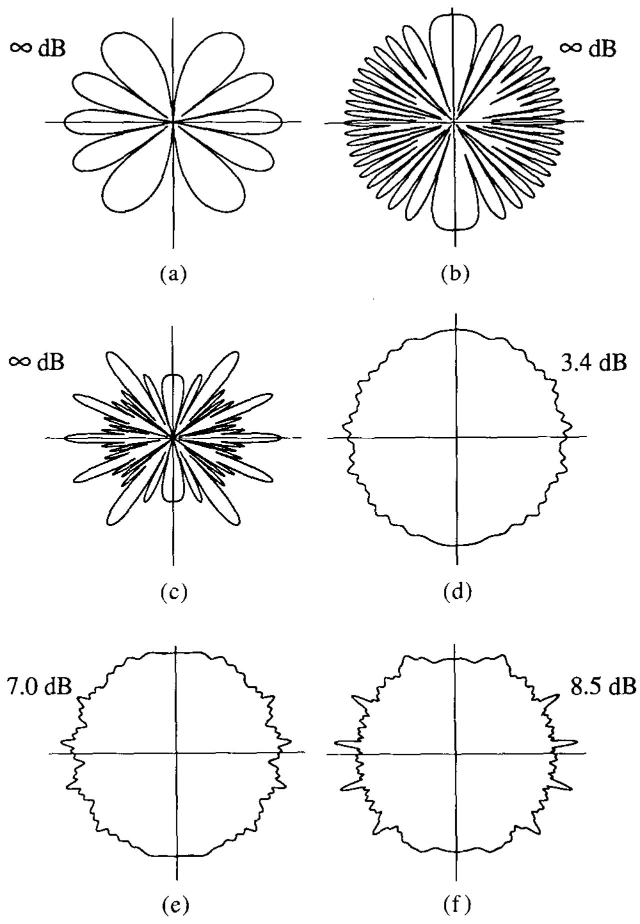


Fig. 30. Comparison of magnitude polars for all analyzed arrays. Polars were all run at a frequency of 10 Hz and a working distance of 20 units. Peak-to-peak polar ripple is listed on each plot. (a) Two-source equal-level equal-polarity equal-spaced array of 0.25-unit spacing. (b) Two-source equal-level equal-polarity equal-spaced array of 1.0-unit spacing. (c) Five-source equal-level equal-polarity equal-spaced array with 1.0-unit center-to-center length. (d) Five-source Bessel array with 1.0-unit center-to-center length. (e) Seven-source Bessel array with 1.5-unit center-to-center length. (f) Nine-source Bessel array with 2.0-unit center-to-center length. Superiority of five-source Bessel array (d) is very clear.

5.4 Frequency Response Comparison

Fig. 31 displays a comparison of off-axis frequency responses for all the analyzed arrays. The response curves were all run at the same off-axis angle (+45°) and working distance (20 units) and covered the same frequency range (0.1–20 Hz). Again, the five-source Bessel array has the smoothest and most extended response.

5.5 Ripple versus Frequency Comparison

Fig. 32 shows a comparison of the polar peak-to-peak ripple versus frequency for all the analyzed arrays at a working distance of 20 units. The superiority of the five-source Bessel array is again quite clear.

6 CONCLUSIONS

When compared to the other analyzed arrays, the five-source Bessel line array is the clear winner, considering 1) polar response, 2) off-axis frequency re-

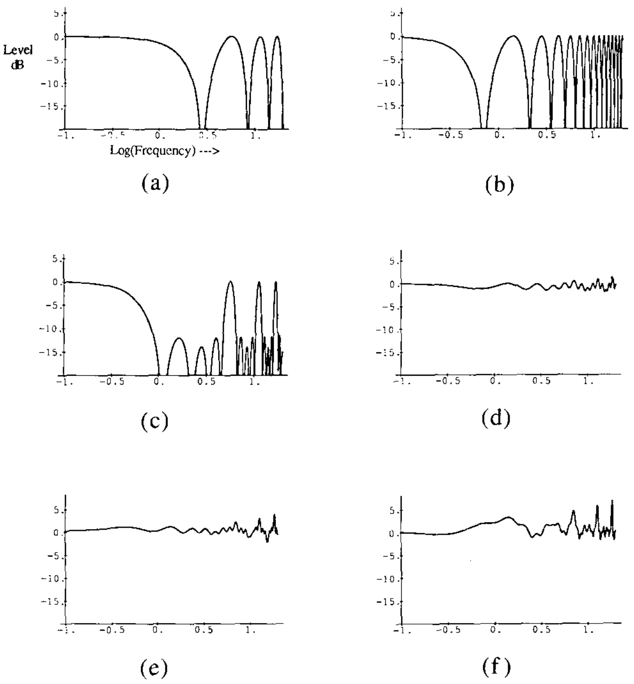


Fig. 31. Comparison of off-axis magnitude frequency responses for all analyzed arrays. Response curves were all run at +45° with a working distance of 20 units, and cover the same frequency range of 0.1 to 20 Hz. Note that log of frequency is indicated (−1 = 0.1 Hz, 0 = 1 Hz, etc.). (a) Two-source equal-level equal-polarity equal-spaced array of 0.25-unit spacing. (b) Two-source equal-level equal-polarity equal-spaced array of 1.0-unit spacing. (c) Five-source equal-level equal-polarity equal-spaced array with 1.0-unit center-to-center length. (d) Five-source Bessel array with 1.0-unit center-to-center length. (e) Seven-source Bessel array with 1.5-unit center-to-center length. (f) Nine-source Bessel array with 2.0-unit center-to-center length. Again, five-source Bessel array has smoothest and most extended response.

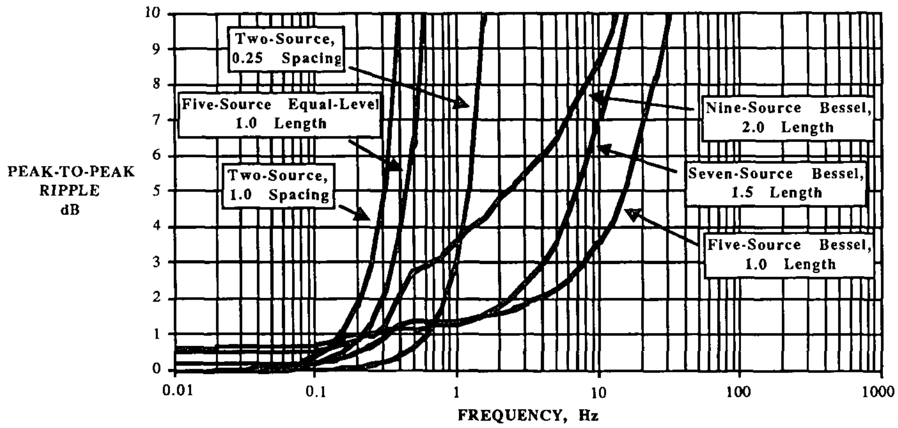


Fig. 32. Comparison of polar magnitude peak-to-peak ripple versus frequency for all analyzed arrays at a working distance of 20 units. Superiority of five-source Bessel is again quite clear. Velocity of propagation 1 unit/s.

sponse, 3) bandwidth of operation, 4) efficiency–bandwidth product, 5) power–bandwidth product, and 6) power–bandwidth product per unit.

Considering the maximum frequency of operation for omnidirectional radiation, at a typical working distance of 20 times the length of the array, the Bessel array outperforms the same-length five-source equal-level equal-polarity equal-spaced array by a factor of 28 and the one-quarter-length equal-level equal-polarity equal-spaced array by a factor of 10. Its power–bandwidth product exceeds that of its nearest competitor, a seven-source Bessel array, by a factor of 2.

The seven- and nine-source Bessel line arrays were found to be effectively unusable due to poor performance, as compared to the five-source Bessel array. Their much lower efficiency, requirement of additional sources, and much lower bandwidth placed them at a severe performance disadvantage.

The Bessel array's singular main problem is its nonlinear phase behavior with direction and frequency. This nonlinear behavior makes it difficult to use the array in conjunction with any other source. Crossing it over to a high-frequency device would be difficult and would require a high slope crossover to minimize off-axis lobing effects in the crossover region. The off-axis phase versus frequency response of the Bessel array is nonminimum phase and exhibits an oscillating phase characteristic. The Bessel array's 45° off-axis group delay versus frequency performance indicates that its time center ranges over a peak-to-peak shift of greater than 25% of the length of the array as the frequency increases.

The Bessel array does not exhibit normal near-field/far-field behavior. Its performance characteristics and high-frequency response get better and better the farther away you are from the array. This is in sharp contrast to the analyzed two- and five-source equal-level equal-polarity equal-spaced arrays, where there was a definite shift from near-field behavior, where the characteristics changed strongly with the working distance, to far-field behavior, where the characteristics changed very little with distance.

An analysis was not done on the 25 (5 × 5)-element planar (panel) source. Presumably the strong performance advantages of the five-source Bessel line array would carry over to this configuration.

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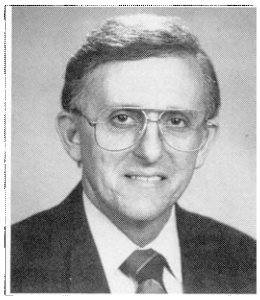
THE AUTHOR



D. B. (Don) Keele, Jr., was born in Los Angeles, CA, in 1940. After serving in the U.S. Air Force for four years as an aircraft electronics technician, he attended California State Polytechnic University at Pomona, from which he graduated with honors and B.S. degrees in both electrical engineering and physics. Mr. Keele worked as an audio systems engineer for Brigham Young University in Provo, Utah, where he received his M.S. degree in electrical engineering in 1975 with a minor in acoustics.

From 1972 to 1976, Mr. Keele worked at Electro-Voice, Inc. in Buchanan, MI, as a senior design engineer in loudspeakers, concentrating on high-frequency horns and low-frequency vented-box loudspeaker systems. He is the primary designer of their HR series of constant-directivity horns on which he holds the patent. For one year, starting in 1976, he worked for Klipsch and Associates in Hope, AR, as chief engineer involved in the company's commercial line of loudspeakers. From 1977 to 1984, he was with JBL, Inc, in Northridge, CA, as a senior transducer engineer working on horn and monitor loudspeaker system design. He also holds two patents on JBL's Bi-Radial series of constant-directivity horns.

Mr. Keele was employed by the Techron Division, Crown International, Elkhart, IN from 1984 to 1989, where he was manager of software development and responsible for the TEF System 12 time-delay spectrometry analyzer software. While at Techron, he was the programmer for two software packages for the TEF System: EasyTEF, a program for doing general-purpose TDS measurements; and TEF-STI, a program for measuring speech intelligibility.

Since October of 1989, he has been a self-employed independent consultant with his own company, DBK Associates, working primarily for *Audio* magazine, Diamandis Communications, as their Senior Editor in charge of loudspeaker reviews. He also is a consultant to Crown working with advanced TEF system development.

A member and fellow of the Audio Engineering Society, Mr. Keele has presented and published a number of papers on loudspeaker design and measurement methods, among them the paper for which he won the AES Publications Award, "Low-Frequency Loudspeaker Assessment by Nearfield Sound-Pressure Measurement" (*J. Audio Eng. Soc*, vol. 22, p. 154 (1974 Apr.). He is a frequent speaker at AES section meetings and workshops, is a member of several AES committees, and is on the AES *Journal* review board.